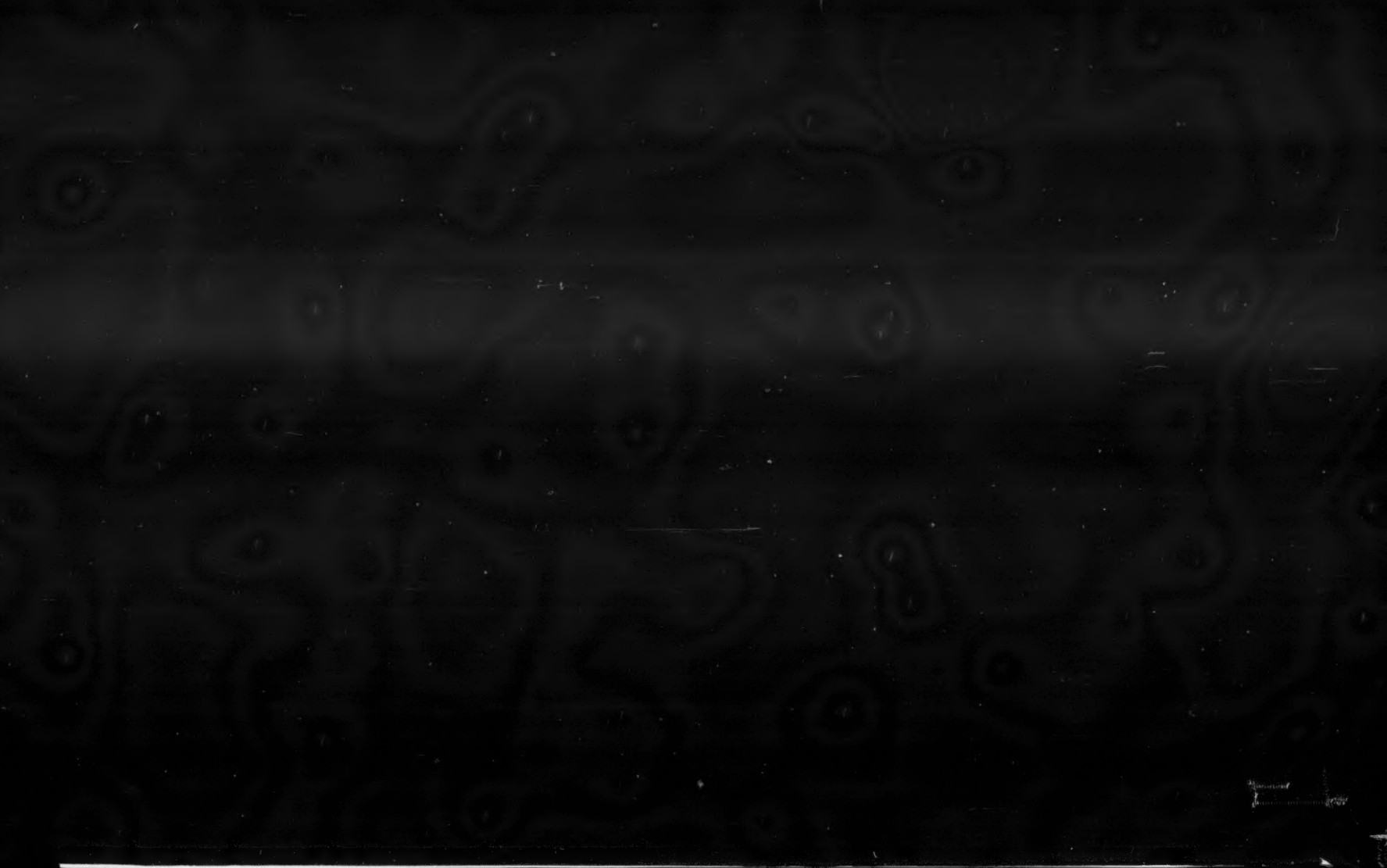

OBJECTIVES IN THE TEACHING OF COLLEGE MATHEMATICS*

F. S. NOWLAN, University of Illinois

1. Introduction. Any constructive study of objectives in the teaching of mathematics must presuppose an agreement as to what is meant by mathematics. In this paper, we concern ourselves with mathematics in the sense of mathematical thinking and not as an inclusive term applying to the vast body of mathematical truths; also we are not thinking primarily of the mastery of rules and devices for the manipulation of numbers and algebraic expressions. What, then, is mathematics in this sense and what is its value?

It is a truism to state that mathematical thinking is deductive reasoning, but we can profitably examine this statement. Deductive reasoning is based upon assumptions, and these apply to what, for lack of a better word, we will designate elements. The elements may, in part, be accepted without definition, for example, our concept of point in geometry. Any mathematical development begins by assuming that the elements under consideration satisfy certain conditions, that is, are subject to certain rules of behavior. Mathematics consists in the logical deduction of the necessary consequences of the initial assumptions.

It may happen that the assumptions clash, or lead to conclusions which contradict established or accepted theory. One may then modify the assumptions, or start with a new set, or, in the latter contingency, one may be radical and hold to the initial assumptions and ignore the accepted theory. This radical action has occurred frequently in the development of science and has made scientific progress possible.

The procedure that we have outlined is basic in mathematical thinking, and, although not always followed, is useful in every field of human endeavor.

Unfortunately, mathematics, in this sense, will fail to bring about a Utopia. The initial assumptions may include the doctrine that the greatest good results from self-advancement, or that one's chief mission in life is the dissemination of certain creeds or ideologies, or the retention of certain creeds or ideologies, and that the end justifies the means. The individual may then reason logically and honestly, that is, in agreement with his assumptions, but the result may be a Hitler, or worse, whose success or failure, proves or disproves (according to accepted standards of proof) his theories.

All this goes to show that mathematical thinking, in itself, is no cure-all for human ills, but it is essential if these ills are to be alleviated. To repeat, mathematical thinking merely means the study of the necessary consequences of initial assumptions which motivate courses of action. As such it applies even to the social and economic sciences. It is non-moral in character and has no concern with variable standards of rightness or wrongness in behavior.

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2. Objectives in teaching mathematics. By now you are wondering what this has to do with objectives in the teaching of college mathematics. Let me explain, and in doing so you will understand that I am merely presenting my personal views and reactions to conditions which are fairly common in our colleges and universities. Also, I would emphasize at this point that I do not underestimate the value of practical applications in the teaching of mathematics, nor the necessity for the mastery of mechanical skills in numerical and algebraic operations. I assume that these are taken care of, as a matter of course, in our instruction.

There are two main aims in the teaching of college mathematics, whether to liberal arts students or to students of engineering. These, in order of importance, are: (1) Training in precise thinking and a grasp of basic principles. (2) The acquisition of information and a mastery of certain technical skills.

It appears, however, that, in general, a disproportionate stress has been placed upon the second objective to the almost total neglect of the first. The excuse has been that the time at the instructor's disposal is limited and that present day students are poorly prepared and intellectually immature.

In answer to these objections, we point out that mechanical work that is not based upon an understanding of principles has little value, practical or otherwise, and that a student can progress rapidly, and without mental strain, only when he works understandingly.

It is true that in every student body there are those who are incapable of logical thinking but we should hardly permit them to determine our standards. Also, there are those who are so gifted intellectually that they will progress under any system and in spite of handicaps of instruction and textbooks. It happens, however, that the majority of students are in a middle class; they need guidance and can profit from guidance. They are capable of logical thinking but, in many cases, have never undergone that experience. These students constitute the main reason for instructors in our colleges.

One of the first duties of the instructor should be to purge the student's mind of the belief that mathematics is a purely mechanical art and that it consists in the formal application of rules that must be memorized. This concept of mathematics, which is fairly common among both students and instructors, stultifies the intellect and the imagination.

To digress, I recall the case of an ex-marine, in a navy V-12 course, a young man of more than average perseverance and intelligence. This student was asked to show that three points A, B, and C were collinear. He found that the lengths of AB, BC, and AC were $\sqrt{13}$, $\sqrt{13}$, and $\sqrt{52}$, respectively, and remarked, "We must prove $\sqrt{52}=2\sqrt{13}$." Then follows the deduction:

$$\sqrt{13} = \sqrt{4+9} = 2+3 = 5,$$

$$\sqrt{52} = \sqrt{16+36} = 4+6 = 10,$$

but

$$10 = 2 \times 5,$$

therefore

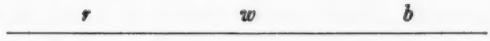
$$\sqrt{52} = 2\sqrt{13}.$$

In doing this, the young man showed imagination and some power of analysis and, in the best tradition of his service, he attained his objective, but he had never learned to associate mathematics and thinking.

3. Some suggestions. How then can we remedy matters? Let me presume to suggest a means. Because I believe that it is only by virtue of an intelligent understanding and grasp of principles that a student can obtain lasting profit from his mathematical studies, and also make progress and acquire speed in his later work, and because I believe that algebra is the study that can best give the student the requisite viewpoint and training, I suggest that more time be devoted to that subject. Not only are we obliged to give the student a new point of view but, frequently, we must eradicate the result of earlier training. This takes time and cannot be accomplished overnight; certainly we can not build a lasting structure upon a superficial foundation. I therefore suggest that the instruction in algebra courses, for the first few weeks, be deliberate, and that the stress, in the first few lessons, be upon the nature of deductive reasoning.

In this connection, the instructor might find profit in reading a work such as the recently published book by B. W. Jones, entitled *Elementary Concepts of Mathematics*. Although serving as an introduction to abstract thinking, the discussion and examples which appear in this book are such as to arouse the interest of the student and stimulate his imagination. Unfortunately, on the other hand, we find influential writers on the teaching of mathematics who seem obsessed with the notion that student interest and mental stimulus can be obtained only by a study of problems which deal with dollars and cents or which concern some form of physical activity. A learning process that is so motivated can scarcely broaden the student's intellectual outlook, and would tend to accentuate his adherence to preconceived notions and personal prejudices. It would be of doubtful cultural value.

The instructor might next take up with his class the growth of the number system and indicate the part that postulates play in its development. The treatment would be abstract since it deals with numbers which are abstractions, but it could be presented so as to capture the students' interest. One of the difficulties at this stage of the work is the necessity of convincing the student that rules of operation must be reexamined with every extension of the number system. This can be done, and the student's interest captured, by an illustration somewhat similar to the following: Consider three colored beads, red, white, and blue, upon a wire in the order



An interchange of the beads to produce a new order, say,

$$w \qquad b \qquad r$$

is called a substitution and may be designated by the symbol

$$s_1 = \begin{pmatrix} r & w & b \\ w & b & r \end{pmatrix},$$

which indicates the replacement of r by w, w by b, and b by r. In case we are dissatisfied with the new color scheme, we may continue with another substitution, say:

$$s_2 = \begin{pmatrix} w & b & r \\ w & r & b \end{pmatrix}$$

to produce the order

$$w \qquad r \qquad b$$

We then observe that the single substitution,

$$\begin{pmatrix} r & w & b \\ w & r & b \end{pmatrix},$$

is the resultant, or equivalent, of the two substitutions s_1 and s_2, in the indicated order. For lack of a better name, we call the resultant substitution the product of s_1 and s_2, in that order, and we agree to represent it by the notation s_1s_2. We also call the process of forming the product multiplication. In doing all this we have used the words product and multiplication in a new sense; that is, we have extended the meaning of these words. It is then fairly obvious that rules for multiplication which were established in our earlier work, say for positive numbers, will not carry over automatically to the multiplication of substitutions. In fact, upon trial, we obtain for the substitution s_2 followed by s_1, the product

$$s_2s_1 = \begin{pmatrix} r & w & b \\ r & b & w \end{pmatrix}.$$

Thus it follows for substitutions that

$$s_1s_2 \neq s_2s_1,$$

whereas, in our earlier work with positive numbers,

$$s_1s_2 = s_2s_1.$$

An illustration of this sort should prepare for a discussion of the problems that attend any extension of the number system. It should also suggest that if,

by definition, we introduce a new type of number, we should never assume that rules, which were not included in our definitions, apply merely because they were established for more elementary types of numbers. We must inquire into the rules of operation every time we extend the number system.

All this means that we propose beginning the study of college algebra with an attempt to indoctrinate the student in the rudiments of abstract thinking. This has the advantage that the student becomes conscious of the postulational character of mathematics and learns at an early stage to refer back to his assumptions before arriving at conclusions. In other words, he gets training in arriving at conclusions which are not based on prejudice, or preconceived notions, or on the word of some authority. He must think for himself, and work from first principles.

At a later time, the introduction of complex numbers into the number system furnishes another excellent opportunity for training in deductive reasoning.

We have now completed the introductory stage in the study of college algebra. From here on, our remarks apply equally well to any course in college mathematics and merely suggest how the instructor may attain the objective that the student shall understand his mathematics and shall be capable of independent thinking. Of course, as mentioned at the outset, we assume throughout that the student shall receive the usual instruction and drill in manipulative processes, except that this shall not dominate the picture. With all this in mind, it is most important that the student shall be required to express himself precisely, and without ambiguity, in speech, in symbolism, and in writing, especially in the latter since while writing he has more opportunity for deliberation. Information which one cannot impart to another has little intrinsic value. Furthermore, a person's inability to express his ideas clearly is indicative of confusion in thinking, and the converse is equally true.

The student must be aware of the exact meaning of the notation which he employs, and this, of course, involves an awareness of any restriction upon the interpretation of notation. I have in mind, for instance, the symbol $\sqrt{a}$, in which a is positive, which by universal agreement represents the positive square root of a. How many students realize that this agreement will compel him to write

$$\sqrt{x^2} = -x,$$

when x represents the abscissa of a point in the second quadrant? In any case, an example of this sort will prove useful. Also, in this connection, the student should be able to locate the fallacy in the statement

$$1 = \sqrt{1} = \sqrt{i^4} = i^2 = -1.$$

The same necessity for proper interpretation arises in the evaluation of the length of arc in the case of the astroid

$$x^{2/3} + y^{2/3} = a^{2/3}.$$

The usual formula gives the integral

$$\int \sqrt{\frac{a^{2/3}}{x^{2/3}}}\, dx.$$

The student should be able to explain why

$$\int_{-a}^{a} \frac{a^{1/3}}{x^{1/3}}\, dx$$

gives zero and not the length of the portion of the astroid above the x-axis. The difficulty is not in the improper integral but in failing to observe that $-a^{1/3}/x^{1/3}$ is the positive square root for (x, y) a point in the second quadrant.

Since writing the above, I have seen a reference to trigonometric identities in a recent number of a leading mathematical journal. The statement was made: "As a matter of fact, both members of the identity will tolerate any fundamental process, except, and here the caution must be flashed, the extraction of roots. For example,

$$\sin^2(-x) = 1 - \cos^2 x, \qquad \text{for any } x.$$

Extracting roots

$$\sin(-x) = \sqrt{1 - \cos^2 x},$$

$$-\sin x = \sqrt{1 - \cos^2 x}."$$

The author seems to labor under the confusion that we have just pointed out. He should have written

$$\pm \sin x = \sqrt{1 - \cos^2 x},$$

with the proviso that the sign plus or minus must be chosen so as to make the expression on the left side positive. For example, if

$$x = 240°,$$

he would have the true relation

$$\sin(-240°) = -\sin 240° = \sqrt{1 - \cos^2 240°}.$$

Another example in interpretation, one which involves a restriction in a differentiation, is the substitution

$$x = a \sin \theta$$

in the integral

$$\int_{-a}^{a} \sqrt{a^2 - x^2}\, dx.$$

Why must the limits in the resulting integral read from $-\pi/2$ to $\pi/2$, rather than from $3\pi/2$ to $\pi/2$? Attention to such details can mean the difference between the student working mechanically or intelligently.

The word "equals" is one of the important technical terms in elementary mathematics. The student must employ this word in its proper sense and also be aware of the fact that an equality sign is not a punctuation mark. The instructor should not accept the statement: "The equation equals," or, "The formula equals." Quite regardless of the numerical result that he obtains, a student should not receive a passing grade on the solution of a problem that concerns the speeds of two airplanes if he starts out: "Let x equal the first airplane."

I have seen these statements in recent student work, and also the following which illustrate additional misuse of the equality sign:

(1)
$$2x - 3y = 7$$
$$= 2x - 3y - 7 = 0.$$

(2)
$$x/2 + y/3 = 1$$
$$= 3x + 2y = 6.$$

(3) In finding the seventh term in the expansion of $(3x-2y)^{10}$:

$$(3x - 2y)^{10} = 7\text{th term} = etc.$$

(4) In finding the foci of the ellipse

$$16x^2 + 25y^2 = 400:$$

$$F = ae = \sqrt{a^2 - b^2} = \sqrt{25 - 16} = 3 = F_1(3, 0) \quad \text{and} \quad F_2(-3, 0).$$

(5) Also,

$$\frac{27.8 \times .0364}{.456} = \frac{\log 27.8 + \log .0364}{\log .456}$$
$$= \frac{1.4440 + 8.5611 - 10}{9.6590 - 10} = \frac{10.0051 - 10}{9.6590 - 10}$$
$$= .3461 = 2.218 \text{ (the correct answer to the problem).}$$

I suggest that to help the student appreciate the fault in work such as the last, the instructor might assign a simple problem of the type: Determine a number x which is less by 3 than the product of 6 and 7. He could then submit to the class, for their approval or disapproval, the solution.

$$x = 6 + 7 \div 3 = 42 \div 3 = 39.$$

If they disapprove, he could exhibit the logarithmic computation and make comparisons. If this makes no impression, the instructor may conclude that the student is not a prospective mathematical genius.

One final example: The student of college algebra is likely to believe that

polynomial equations in x and y can be combined freely by addition, subtraction, and by the substitution of the one into the other and that, provided there have been no numerical errors, the values that are obtained for x and y constitute solutions. The student might, with profit, consider this procedure as applied to the equations

$$(a)\quad (x^2+y^2+x)^2 = 9(x^2+y^2) \qquad \text{and} \qquad (b)\quad x^2+y^2 = 1.$$

The substitution of (b) into (a) gives

$$(x+1)^2 = 9, \qquad x+1 = \pm 3, \qquad x = 2, \qquad -4.$$

If he substitutes 2 for x in (a), he gets the equation

$$(y^2+6)^2 = 9(y^2+4),$$

or,

$$y^2(y^2+3) = 0,$$

which has the real solution

$$y = 0.$$

The student would likely conclude that (2, 0) is a common solution of the given equations and so a common point of their graphs. It happens, however, that the graphs have no point in common and the equations no common solution.

It is desirable that the student should realize that the propriety of combining the equations as we have done is based on an assumption that there is a common pair of values for x and y which satisfy the two equations and that, as a consequence of such an assumption, even when the numerical work is known to be correct, the values that have been obtained for x and y must be checked in both equations.

The student should learn to employ words in their correct sense and so as to convey his meaning without ambiguity, and he must realize that symbolism constitutes a shorthand notation and must be employed correctly if it is to convey information. He should be expected, in his various written tests, to give precise definitions of terms that are employed in his studies and in doing this and in proving principles he should be encouraged to employ, so far as possible, his own phrasing rather than that of the instructor, or the text, verbatim. It would be interesting, and perhaps surprising, to learn the small percentage of those in classes in college mathematics who have any clear conception of the meaning of the terms which they employ in their daily work, words and phrases such as rational number, polynomial in x, polynomial in x of the nth degree, sum of an infinite series, convergent series, *etc.* Of course, in fairness, we should not place the entire blame for the student's inability to express himself upon his early training in mathematics. His instruction, or lack of instruction, in English must, in part, be held responsible.

We find that in many cases the student is adverse to reading his textbooks and is inclined to depend too much upon the instructor. It would help matters if he were not only required to read the text but were held responsible for interpreting assignments from it to the class.

From all that has been said, one might infer, and correctly so, that the speaker has little sympathy with many of the so-called objective tests and multiple choice questions in examinations. These appear, to him, to defeat the very aim of education, since they require little more of the student than the ability to mark the spot with an x, and surely there is more to a measure of educational attainment than the winning of credits, especially when we simplify the process to this extent.

I realize that these remarks are out of step with some present-day educational theory. However, when, in educational journals, I find the emphasis on mathematical textbooks (and I quote) "being easy," "meaningful to the student," "not a fusion of abstract mathematics," and find references to "hatred towards mathematics" engendered by its difficulty, and this in the very elementary books, then I admit that I am not greatly concerned if I seem out of step with certain theorists. It happens that we are not traveling in the same direction. Also, unlike these people, I am not prepared to grant that American youth is less intelligent than the youth of other countries. It seems desirable to me that a textbook should be sufficiently difficult to require some mental effort on the part of the student. Otherwise, it can scarcely serve as a means of mental stimulus and intellectual development. Furthermore, I feel that we should not be disturbed by adverse criticisms of mathematics which may come from the man on the street, or from non-mathematicians. Their views on mathematics should carry the same weight as their views on relativity or on the technique of atomic research. In general, the so-called popular treatments of mathematics and science are a doubtful blessing and are likely to give rise to misleading concepts.

4. Summary. In summary: We have indicated that mathematics should be taught with the emphasis upon the thinking process. However, we have warned that this statement must not be interpreted as discounting the value of applied mathematics. Applied mathematicians must think. The statement merely implies that there are social values in the study of mathematics, and that these are as important as its applications in industry and in science.

We have shown that proper conclusions are inevitable in a process of reasoning and are a necessary consequence of the initial assumptions, but we have given no recipe for the selection of these assumptions. It is unfortunate, but true, that the initial assumptions may generate intolerance and bigotry, and the danger to our social and economic life is the fact that indoctrination into these premises takes place at an intellectually immature age. In this way they become a part of one's being, and constitute preconceived notions, or prejudices, which limit tolerance. They are accepted as realities and are not looked upon as assumptions.

A social value of the postulational approach to the study of mathematics is that it presupposes an examination and study of the premises. The student comes to realize that in a given study there are alternative sets of premises, all equally possible and proper. Thus, for example, in building an algebra (in the sense of linear algebras) he may start with any one of various sets of postulates, all equally possible and proper, yet one generates an algebra which is rich, and another barren, in its possibilities. The abstract treatment of algebra assures that the student will develop the habit of considering and questioning sets of postulates and this without bias in favor of one set or another. It is then inevitable that he shall acquire habits of reflection and judgment, and that these will affect his reactions to the problems of everyday life. They will make him more tolerant of opinions with which he disagrees.

We are living in a remarkable but dangerous age, one in which new and powerful forces, physical and spiritual, have been let loose. The individual is faced with a greater challenge, and a greater opportunity, than perhaps at any other time in world history. The forces, some good and some evil, must be met with understanding. This calls for independent thinking, sound judgment, and intelligent leadership. It follows that teachers of college mathematics hold a key position. It is their obligation to train our future leaders in habits of sound thinking, in clearness of vision and expression, and in tolerance, which is a by-product of mathematical thinking. However, to do this, teachers must shift the emphasis from the mechanics of mathematics to its understanding. In doing all this, teachers of mathematics can play a part in moulding world destiny.

ON SUMS INVOLVING BINOMIAL COEFFICIENTS

DAVID DICKINSON, University of Michigan*

1. Preliminary concepts. Let n, a, b, and c be integers. Let the binomial coefficients be defined as

$$\binom{n}{b} = \begin{cases} \dfrac{n!}{(n-b)!\,b!} & \text{for } n \geqq b \geqq 0; \\ 0 & \text{otherwise.} \end{cases}$$

We will evaluate the finite sum

$$\sum_k \binom{n+ak}{b+ck}, \tag{1}$$

where a summation over k will always signify a summation over all integral k.

* This paper was completed while the author was under contract to the Office of Naval Research.

By the determinant of (1), we shall mean $nc-ab$.

Let the Pascal triangle be mapped onto the Cartesian plane by identification of the binomial coefficient $\binom{x}{y}$ with the point (x, y). All of the summands of (1) will lie on a line,

$$y = \frac{c}{a}x - \frac{(nc - ab)}{a}, \tag{2}$$

provided a is not zero. The case $a=0$ will be considered separately.

From the identity,

$$\binom{n}{b} = \binom{n}{n-b},$$

the sum (1) may also be written

$$\sum_k \binom{n + ak}{(n-b) + (a-c)k}. \tag{3}$$

The line passing through the summands of (3) has the slope $(a-c)/a$. Of the two slopes, c/a and $(a-c)/a$, one is greater than unity and the other is negative. (Otherwise, the sum would be infinite.) Therefore, the finite sum may be written in the form (1) with positive slope c/a.

Changing the signs of both a and c in (1) is equivalent to a substitution of $-k$ for k. But since the summation is over all k, positive and negative, such a substitution does not change the value of the sum. Hence we may assume that in (1), $c>a>0$.

2. The case $(a, c)=1$. The restriction on (1) that $(a, c)=1$ implies that all the integral points of (2) correspond to summands of (1), and hence that such a sum is uniquely specified by the equation of its corresponding line.

Consider the set of sums that can be written in the form (1) with the same a and c. Their corresponding lines, from (2), are parallel. Since a and c are relatively prime, the determinants of these sums form precisely the set of all integers. From (2), the y-intercept of a line is the quotient of the determinant of the corresponding sum and $-a$. Hence these parallels are equally spaced.

Let S_m denote the sum on the line whose determinant is m. Consider the sequence,

$$\cdots S_{-1}, S_0, S_1, \cdots S_m, \cdots . \tag{4}$$

We have $S_m=0$ for $m<0$ since such lines, having slopes greater than one and positive y-intercepts, do not meet the non-zero portion of the Pascal triangle.

It follows from the fundamental identity of the Pascal triangle, namely,

$$\sum_k \binom{n + ak}{b + ck} = \sum_k \binom{n - 1 + ak}{b - 1 + ck} + \sum_k \binom{n - 1 + ak}{b + ck}, \tag{5}$$

that the sequence (4) has the recurrence relation

$$S_m = S_{m-c+a} + S_{m-c}, \qquad (m \neq 0). \tag{6}$$

Let α_r $(r=0, 1, \cdots, (c-1))$ be the roots of

$$1 = x^{-c+a} + x^{-c}, \tag{7}$$

the characteristic equation of (6), or of the polynomial equation with the same roots,

$$x^c - x^a - 1 = 0. \tag{8}$$

Then

$$S_m = \sum_{r=0}^{c-1} d_r \alpha_r^m, \tag{9}$$

where the d_r $(r=0, 1, \cdots, (c-1))$ are arbitrary constants, satisfies (6), as can be verified by direct substitution in (6) and comparison with (7). In order that (9) should give the terms of (4), the d_r must satisfy the following set of equations:

$$\begin{aligned}
S_0 &= 1 = d_0 + d_1 + \cdots + d_{c-1} \\
S_{-1} &= 0 = d_0\alpha_0^{-1} + d_1\alpha_1^{-1} + \cdots + d_{c-1}\alpha_{c-1}^{-1} \\
S_{-2} &= 0 = d_0\alpha_0^{-2} + d_1\alpha_1^{-2} + \cdots + d_{c-1}\alpha_{c-1}^{-2} \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
S_{-(c-1)} &= 0 = d_0\alpha_0^{-(c-1)} + d_1\alpha_1^{-(c-1)} + \cdots + d_{c-1}\alpha_{c-1}^{-(c-1)}.
\end{aligned}$$

The Vandermonde determinant,

$$|\alpha_i^{-j}| = \prod_{i>j} (\alpha_i^{-1} - \alpha_j^{-1}), \quad i, j = 0, 1, \cdots, (c-1),$$

is not zero since, as will be shown later, (8) has no multiple roots. The determinant formed by replacing the column of $|\alpha_i^{-j}|$ involving α_r by the S_{-j}'s reduces immediately to a product involving another Vandermonde determinant:

$$(-1)^r \alpha_0^{-1} \cdots \alpha_{r-1}^{-1} \cdot \alpha_{r+1}^{-1} \cdots \alpha_{c-1}^{-1} \prod_{i>j} (\alpha_i^{-1} - \alpha_j^{-1}),$$

$$i, j = 0, 1, \cdots, (r-1), (r+1), \cdots, (c-1).$$

By Cramer's rule, we have

$$d_r = \frac{(-1)^r \alpha_0^{-1} \cdots \alpha_{r-1}^{-1} \cdot \alpha_{r+1}^{-1} \cdots \alpha_{c-1}^{-1}}{(-1)^{c-r-1} \prod (\alpha_r^{-1} - \alpha_j^{-1})} = \frac{\alpha_r^{c-1}}{\prod (\alpha_r - \alpha_j)}$$

where $j=0, 1, \cdots, (r-1), (r+1), \cdots, (c-1)$. But the denominator of the last expression is the derivative of the left member of (8) evaluated at $x=\alpha_r$. Hence

$$d_r = \frac{\alpha_r^{c-1}}{c\alpha_r^{c-1} - a\alpha_r^{a-1}},$$

and, therefore,

$$S_m = \sum_{r=0}^{c-1} \frac{\alpha_r^m}{c - a\alpha_r^{a-c}}.$$

3. The general case. In order to sum (1) when a and c are not relatively prime, we must first generalize the above result in a slightly different direction.

Consider the sum

$$S_m(t) = \sum_k \binom{n+ak}{b+ck} t^{b+ck}, \tag{10}$$

where $m = nc - ab$, $|t| \leqq 1$, and a and c have the same restrictions as before. After multiplying each element $\binom{x}{y}$ of the Pascal triangle by t^y, we have the relation, derived from (5),

$$\sum_k \binom{n+ak}{b+ck} t^{b+ck} = t\sum_k \binom{n-1+ak}{b-1+ck} t^{b-1+ck} + \sum_k \binom{n-1+ak}{b+ck} t^{b+ck},$$

which may be written

$$S_m(t) = tS_{m-c+a}(t) + S_{m-c}(t), \qquad (m \neq 0). \tag{11}$$

We must show that the characteristic equation of (11),

$$1 = tx^{-c+a} + x^{-c},$$

or the equivalent equation,

$$x^c - tx^a - 1 = 0, \tag{12}$$

has no multiple roots. Taking the derivative of the left member of (12), removing the roots $x=0$ (since they obviously are not roots of (12)), and eliminating x between the result and (12), we obtain

$$\left(\frac{at}{c}\right)^{c/(c-a)} - t\left(\frac{at}{c}\right)^{a/(c-a)} = 1$$

or

$$\left(\frac{at}{c}\right)^{c/(c-a)} = \frac{a}{a-c}.$$

But this conflicts with our original assumption that $c > a > 0$. Hence (12) and (8) have no multiple roots.

The sum (10) is then given by

$$S_m(t) = \sum_{r=0}^{c-1} \frac{\alpha_r^m}{c - at\alpha_r^{a-c}},$$

where the α_r $(r=0, 1, \cdots, (c-1))$ are the roots of (12).

Consider the expression

$$\frac{1}{d} \sum_{s=1}^{d} e^{(2\pi iks)/d},$$

where d is an integer greater than one. If k has d as a factor, the sum is one; otherwise the sum is zero. Hence

$$\frac{1}{d} \sum_k \left\{ \binom{n+ak}{b+ck} \sum_{s=1}^{d} e^{(2\pi iks)/d} \right\} = \sum_k \binom{n+adk}{b+cdk}. \tag{13}$$

But the left member of (13) also reduces to

$$\frac{1}{d} \sum_{s=1}^{d} \left\{ e^{(-2\pi ibs)/dc} \sum_k \binom{n+ak}{b+ck} e^{(2\pi i)(b+ck)s/dc} \right\}. \tag{14}$$

Hence,

$$\sum_k \binom{n+adk}{b+cdk} = \frac{1}{d} \sum_{s=1}^{d} \omega^{-bs} S_m(\omega^s), \qquad \omega = e^{(2\pi i)/dc}. \tag{15}$$

We can now also consider the case where a is zero, that is,

$$\sum_k \binom{n}{b+dk}. \tag{16}$$

Suppose $d=1$, then (16) is merely 2^n. Assume $d>1$. Then, setting $c=1$ and $k'=b+k$, we have, from (14) and (15),

$$\sum_k \binom{n}{b+dk} = \frac{1}{d} \sum_{s=1}^{d} \left\{ e^{(-2\pi ibs)/d} \sum_{k'=-\infty}^{+\infty} \binom{n}{k'} e^{(2\pi i)k's/d} \right\}$$

or, the known result,

$$\sum_k \binom{n}{b+dk} = \frac{1}{d} \sum_{s=1}^{d} \omega^{-bs}(1+\omega^s)^n, \qquad \omega = e^{(2\pi i)/d}. \tag{17}$$

It is interesting to note that since the sums

$$\sum_k \binom{n+k}{b+2k}$$

form the Fibonacci series, all of our series of sums may be considered as generalizations of the Fibonacci series.

AN EXTENSION OF FERMAT'S THEOREM

A. A. TRYPANIS, Athens, Greece

In Fermat's Theorem it is stated that if p is a positive, rational, odd prime integer and if a is a rational integer such that

$$a \not\equiv 0 \pmod{p},$$

then

$$a^{p-1} - 1 \equiv 0 \pmod{p}. \tag{1}$$

From this theorem, it follows immediately that

$$\begin{aligned} a^{p(p-1)} - 1 &\equiv 0 \pmod{p^2}, \\ a^{p^2(p-1)} - 1 &\equiv 0 \pmod{p^3}, \\ a^{p^3(p-1)} - 1 &\equiv 0 \pmod{p^4}, \\ &\cdots\cdots\cdots\cdots, \\ a^{p^{n-1}(p-1)} - 1 &\equiv 0 \pmod{p^n}, \end{aligned} \tag{2}$$

where n may possess any positive, rational, integral value.

Let this sequence of congruences be considered in an opposite sense to that in which it has been written above; that is, let us write

$$\begin{aligned} a^{p^{n-1}(p-1)} - 1 &\equiv 0 \pmod{p^n}, \\ a^{p^{n-2}(p-1)} - 1 &\equiv 0 \pmod{p^{n-1}}, \\ a^{p^{n-3}(p-1)} - 1 &\equiv 0 \pmod{p^{n-2}}, \\ &\cdots\cdots\cdots\cdots, \\ a^{p^2(p-1)} - 1 &\equiv 0 \pmod{p^3}, \\ a^{p(p-1)} - 1 &\equiv 0 \pmod{p^2}, \\ a^{p-1} - 1 &\equiv 0 \pmod{p}. \end{aligned}$$

It will be shown that within the domain of algebraic integers this "downward sequence" can be extended as follows:

$$\begin{aligned} a^{(p-1)/p} - 1 &\equiv 0 \pmod{p^{1/p}}, \\ a^{(p-1)/p^2} - 1 &\equiv 0 \pmod{p^{1/p^2}}, \\ a^{(p-1)/p^3} - 1 &\equiv 0 \pmod{p^{1/p^3}}, \\ &\cdots\cdots\cdots\cdots \\ a^{(p-1)/p^n} - 1 &\equiv 0 \pmod{p^{1/p^n}}. \end{aligned} \tag{3}$$

Let $K(Z)$ be the cyclotomic field generated by a p^{n+1}th root of unity

$$Z = e^{2i\pi/p^{n+1}}; \tag{4}$$

then if E is the algebraic number defined by

$$p = E(1 - Z)^{p^n(p-1)} \tag{5}$$

it may be shown that E is a unit. For on setting up the irreducible equation satisfied by $1-Z$ it is found that $N(1-Z)=p$, while

$$(1 - Z)^{p^n(p-1)} = p\overline{E} \tag{6}$$

where $\overline{E}$ is an integer of $K(Z)$. Hence, taking norms in $K(Z)$, $p^{p^n(p-1)} = p^{p^n(p-1)}N(\overline{E})$, and $N(\overline{E})=1$. Therefore $\overline{E}$, and likewise its reciprocal E, is a unit.

Let

$$x = \theta$$

be one of the roots of the irreducible equation

$$x^{p^n} - a = 0; \tag{7}$$

then the p^n conjugate roots of (7) are

$$\theta_j = Z^{p(j-1)}\theta, \qquad j = 1, 2, 3, \cdots, p^n. \tag{8}$$

Thus

$$x^{p^n} - a = \prod_{j=1}^{j=p^n} (x - \theta_j). \tag{9}$$

From the binomial expansion

$$(\theta^{p-1} - 1)^{p^n} = \theta^{p^n(p-1)} - \binom{p^n}{1}\theta^{(p^n-1)(p-1)} + \binom{p^n}{2}\theta^{(p^n-2)(p-1)} - \cdots$$
$$- \binom{p^n}{p^n - 2}\theta^{2(p-1)} + \binom{p^n}{p^n - 1}\theta^{p-1} - 1,$$

it follows that

$$(\theta^{p-1} - 1)^{p^n} \equiv 0 \pmod{p}, \tag{10}$$

because

$$\binom{p^n}{1} \equiv \binom{p^n}{2} \equiv \binom{p^n}{3} \equiv \cdots \equiv \binom{p^n}{p^n - 1} \equiv 0 \pmod{p}$$

and

$$\theta^{p^n(p-1)} - 1 = a^{p-1} - 1 \equiv 0 \pmod{p}.$$

Let

$$(\theta^{p-1} - 1)^{p^n} = B(1 - Z)^{p^n(p-1)}, \tag{11}$$

and let $x=H$ be one of the p^n roots of the equation $x^{p^n}=B$.

From (11) it is evident that a suitable value of H will satisfy the equation

$$\theta^{p-1}-1=H(1-Z)^{p-1}. \tag{12}$$

Thus

$$a^{p-1}-1=\prod_{j=1}^{j=p^n}(\theta_j^{p-1}-1)=\left[\prod_{j=1}^{j=p^n}H_j\right](1-Z)^{p^n(p-1)} \tag{13}$$

and, from (6),

$$a^{p-1}-1=\overline{E}\left[\prod_{j=1}^{j=p^n}H_j\right]p. \tag{14}$$

From (6), it follows that a suitable value

$$E_Z=\sqrt[p^n]{\overline{E}}$$

satisfies the equation

$$(1-Z)^{p-1}=E_Z p^{1/p^n},$$

where the value of E_2 depends on the value assigned to $p^{1/p}$.

From (12), it follows that

$$a^{(p-1)p^n}-1\equiv 0 \pmod{[1-Z]^{p-1}}; \tag{15}$$

hence

$$a^{(p-1)/p^n}-1\equiv 0 \pmod{p^{1/p^n}}. \tag{16}$$

As an example, let $p=3$ and $a=2$. According to (16), for $n=1$, it should be that

$$(\sqrt[3]{2})^2-1\equiv 0 \pmod{\sqrt[3]{3}}.$$

This is so, because

$$(\sqrt[3]{2})^2-1=[\sqrt[3]{(\sqrt[3]{2}-1)^2}]\times\sqrt[3]{3}$$

if real roots are employed.

A SYMMETRICAL NOTATION FOR NUMBERS

C. E. SHANNON, Bell Telephone Laboratories

The possibility of representing real numbers in various scales of notation is well known. Thus, in the scale r an arbitrary positive number b may be expanded in the form,

$$b = \sum_{-\infty}^{N} a_n r^n, \qquad 0 \leqq a_n \leqq r - 1,$$

and represented in the "decimal" notation as $a_N a_{N-1} \cdots a_0 \cdot a_{-1} a_{-2} \cdots$. Negative numbers are represented by prefixing a minus sign to the representation of the corresponding positive numbers. Although it seems unlikely that the scale ten will ever be changed for ordinary work, the use of other scales and systems of notation is still of practical as well as mathematical interest. In some types of computing machines, for example, scales other than ten lend themselves more readily to mechanization.

A slight modification of the ordinary expansion gives a representation for numbers with certain computational advantages. Assuming r to be odd, it is seen easily that any positive or negative number b can be represented as

$$b = \sum_{-\infty}^{N} a_n r^n, \qquad -\frac{r-1}{2} \leqq a_n \leqq \frac{r-1}{2},$$

and we may denote b as usual by the sequence of its digits

$$b = a_N \cdots a_0 \cdot a_{-1} \cdots .$$

Both positive and negative numbers are thus represented by a standard notation without a prefixed sign, the sign being implied by the digits themselves; the number is positive or negative according as the first (nonvanishing) digit is greater or less than zero. Every real number has a unique representation apart from those whose expansion ends in an infinite sequence of the digits $(r-1)/2$ or $-(r-1)/2$, each of which has two representations. If this notation were to be used, a simple notation should be invented for the negative digits which suggested their close relation to the corresponding positive digits. For typographical simplicity we shall here denote the negative digits by placing primes on the corresponding positive digits. The notation for the first nine positive and negative integers with $r = 3, 5, 7, 9$ is as follows:

r	−9	−8	−7	−6	−5	−4	−3	−2	−1
3	1′00	1′01	1′11′	1′10	1′11	1′1′	1′0	1′1	1′
5	2′1	2′2	1′2′	1′1′	1′0	1′1	1′2	2′	1′
7	1′2′	1′1′	1′0	1′1	1′2	1′3	3′	2′	1′
9	1′0	1′1	1′2	1′3	1′4	4′	3′	2′	1′

r	0	1	2	3	4	5	6	7	8	9
3	0	1	11′	10	11	11′1′	11′0	11′1	101′	100
5	0	1	2	12′	11′	10	11	12	22′	21′
7	0	1	2	3	13′	12′	11′	10	11	12
9	0	1	2	3	4	14′	13′	12′	11′	10

In general the negative of any number is found by placing a prime on each unprimed digit and taking it off each primed digit. Arithmetic operations with this system are considerably simplified. In the first place the symmetries introduced by this notation make the addition and multiplication tables much easier to learn. For the scale $r=9$ these tables are, respectively, as follows:

+	4′	3′	2′	1′	0	1	2	3	4
4′	1′1	1′2	1′3	1′4	4′	3′	2′	1′	0
3′	1′2	1′3	1′4	4′	3′	2′	1′	0	1
2′	1′3	1′4	4′	3′	2′	1′	0	1	2
1′	1′4	4′	3′	2′	1′	0	1	2	3
0	4′	3′	2′	1′	0	1	2	3	4
1	3′	2′	1′	0	1	2	3	4	14′
2	2′	1′	0	1	2	3	4	14′	13′
3	1′	0	1	2	3	4	14′	13′	12′
4	0	1	2	3	4	14′	13′	12′	11′

·	4′	3′	2′	1′	0	1	2	3	4
4′	22′	13	11′	4	0	4′	1′1	1′3′	2′2
3′	13	10	13′	3	0	3′	1′3	1′0	1′3′
2′	11′	13′	4	2	0	2′	4′	1′3	1′1
1′	4	3	2	1	0	1′	2′	3′	4′
0	0	0	0	0	0	0	0	0	0
1	4′	3′	2′	1′	0	1	2	3	4
2	1′1	1′3	4′	2′	0	2	4	13′	11′
3	1′3′	1′0	1′3	3′	0	3	13′	10	13
4	2′2	1′3′	1′1	4′	0	4	11′	13	22′

The labor in learning the tables would appear to be reduced by a factor of at least two from the corresponding $r=9$ case in ordinary notation. There is no need to learn a "subtraction table"; to subtract, one primes all digits of the subtrahend and adds. The sign of the difference automatically comes out correct, and the clumsy device of "borrowing" is unnecessary. More generally, to add a set of numbers, some positive and some negative, all are placed in a column without regard to sign and added, *e.g.* ($r=9$):

(1′)		(1)		carried numbers.
1	3′	1′	2	
2′	3	1	4	
4′	1′	2	3	
3	2′	3′	4	
3	0	0	1′	
1′	2′	1	3′	
1′	4	1	0	

This process may be contrasted with the usual method where the positive and negative numbers must be added separately, the smaller sum subtracted from the larger and the difference given the sign of the larger, that is, three addition or subtraction processes and a sign rule, while with the symmetrical system one standard addition process covers all cases. Furthermore, in such a sum cancellation is very common and reduces considerably the size of numbers to be carried in memory in adding a column; this follows from the fact that any digit cancels its negative and these may be struck out from a column without affecting the sum. If all digits are equally likely and independent, the sum in a column will have a mean value zero, standard deviation $\sqrt{p(r^2-1)/12}$ where p is the number of numbers being added, while in the usual notation the mean value is $p(r/2)$ with the same standard deviation.

Multiplication and division may be carried out also by the usual processes, and here again signs take care of themselves, although in these cases, of course, the advantage of this is not so great.

We may note also that in the usual system of notation, when we wish to "round off" a number by replacing all digits after a certain point by zeros, the digits after this point must be inspected to see whether they are greater or less than 5 in the first place following the point. In the former case the preceding digit is increased by one. With the symmetrical system one always obtains the closest approximation merely by replacing the following digits by zeros. Numbers such as $1.444\cdots = 2.4'4'4'\cdots$ with two representations are exactly half way between the two nearest rounded off approximations, and in this case we obtain the upper or lower approximation depending on which representation is rounded off. If we were using this notation, department stores would find it much more difficult to camouflage the price of goods with \$.98 labels.

We have assumed until now that the scale r is odd. If r is even, say 10, a slightly unbalanced system of digits can be used; for example, 4′, 3′, 2′, 1′, 0, 1, 2, 3, 4, 5. The dissymmetry introduced unfortunately loses several of the advantages described above, *e.g.*, the ease of negation and hence of subtraction, and also the round off property.

A more interesting possibility is that of retaining symmetry by choosing for "digits" numbers halfway between the integers. In the case $r=10$ the possible digits would be

$$a_n = \frac{9'}{2}, \frac{7'}{2}, \frac{5'}{2}, \frac{3'}{2}, \frac{1'}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2},$$

and any number b can be expressed as

$$b = \sum_{-\infty}^{N} a_n r^n.$$

In this system the properties of positive-negative symmetry, automatic handling of signs, and simple round off are retained. One curious and disadvantageous feature is that the integers can only be represented as infinite decimals, and this is possible in an infinite number of different ways. For example.

$$0 = \cdot\frac{1}{2}\frac{9'}{2}\frac{9'}{2}\cdots = \frac{1}{2}\cdot\frac{9'}{2}\frac{9'}{2}\frac{9'}{2}\cdots = \frac{1'}{2}\cdot\frac{9}{2}\frac{9}{2}\frac{9}{2}\cdots \text{ etc.}$$

Symmetrical notation offers attractive possibilities for general purpose computing machines of the electronic or relay types. In these machines it is possible to perform the calculations in any desired scale and only translate to the scale ten at input and output. The use of a symmetrical notation simplifies many of the circuits required to take care of signs in addition and subtraction, and to properly round off numbers.

MATHEMATICAL NOTES

EDITED BY E. F. BECKENBACH, University of California, Los Angeles

Material for this department should be sent direct to E. F. Beckenbach, University of California, Los Angeles 24, California.

A NOTE ON AN INEQUALITY FOR INTEGRALS

A. E. TAYLOR, University of California, Los Angeles

This note is concerned with the question as to when the inequality

$$\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|\,dx \tag{1}$$

becomes an equality. We shall consider continuous functions of the real variable x; the function values may be real or complex, or more generally, may lie in a Banach space B. In the latter event the absolute value signs are to be read as norms in the Banach space. It is immaterial whether the interval of integration is finite or infinite, provided the integrals in question exist.

For real continuous functions the equality in (1) holds if and only if $f(x)$ is

either always nonnegative or always nonpositive on the interval (a, b). This familiar theorem may easily be reduced to the theorem that the integral of a continuous nonnegative function is zero only if the integrand is identically zero.

For complex continuous functions the equality in (1) holds if and only if the function values $f(x)$ all lie on a fixed ray emanating from the origin in the complex plane. An equivalent statement of the condition is that all the function values must be nonnegative multiples of a fixed nonzero complex number. This theorem seems intuitively evident when we think of the integrals as sums; it is not as familiar as the corresponding theorem for real functions, however, and the proof is not as simple. We shall give a proof whose form is at once applicable to the case in which the function values lie in a Banach space. The theorem stated at the beginning of the present paragraph is a special case of Theorem 2. For an instance where it is essential to know that the inequality (1) is strict, see Landau [1].

THEOREM 1. *Let B be a Banach space, and let $f(x)$ be continuous at all finite points of the interval $a \leqq x \leqq b$, with values in B. In case the interval is infinite we assume that the integral on the right in* (2) *exists. Then the equality*

$$\left\| \int_a^b f(x)dx \right\| = \int_a^b \|f(x)\|dx \tag{2}$$

holds if and only if for every finite set of distinct points on (a, b) we have

$$\left\| \sum_{i=1}^{n} f(x_i) \right\| = \sum_{i=1}^{n} \|f(x_i)\|. \tag{3}$$

Proof: The sufficiency of condition (3) is easily demonstrated, and we omit the argument. In case the interval (a, b) is infinite we first prove that (3) implies (2) for each finite subinterval of (a, b).

In proving the necessity of (3) we first observe that if (2) holds, then

$$\left\| \int_E f(x)dx \right\| = \int_E \|f(x)\|dx, \tag{4}$$

where E is any subset of (a, b) for which the integrals have a meaning. For, denoting the complement of E in (a, b) by CE, we have

$$\left\| \int_a^b f dx \right\| \leqq \left\| \int_E f dx \right\| + \int_{CE} \|f\|dx,$$

$$\int_a^b \|f\|dx = \int_E \|f\|dx + \int_{CE} \|f\|dx.$$

Thus (2) implies that

$$\int_E \|f\|dx \leqq \left\| \int_E f dx \right\|.$$

Since the reverse inequality also holds, (4) is true.

It suffices to prove (3) for points $x_1, \cdots, x_n$ interior to (a, b), since f is continuous. Suppose then that we are given such points, and let $y_i = f(x_i)$. If $\epsilon > 0$, let E_i be the interval $|x - x_i| \leqq \delta$, where δ is a positive number chosen in such a way that the sets $E_1, \cdots, E_n$ lie in (a, b) and do not overlap, while $\|f(x) - f(x_i)\| < \epsilon$ in E_i. Let $E = E_1 + \cdots + E_n$. Then

$$\int_E f dx = \sum_{i=1}^n \int_{E_i} (f(x) - y_i) dx + \sum_{i=1}^n \int_{E_i} y_i dx,$$

$$\left\| \int_E f dx \right\| \leqq 2n\delta\epsilon + 2\delta \left\| \sum_{i=1}^n y_i \right\|;$$

$$\int_E \|f\| dx = \sum_{i=1}^n \int_{E_i} \|f(x) - y_i + y_i\| dx$$

$$\geqq \sum_{i=1}^n \int_{E_i} (\|y_i\| - \|f(x) - y_i\|) dx$$

$$\geqq 2\delta \sum_{i=1}^n \|y_i\| - 2n\delta\epsilon.$$

From (4) and the foregoing considerations we conclude that

$$\sum_{i=1}^n \|y_i\| \leqq 2n\epsilon + \left\| \sum_{i=1}^n y_i \right\|.$$

It follows that

$$\sum_{i=1}^n \|y_i\| \leqq \left\| \sum_{i=1}^n y_i \right\|;$$

the reverse inequality holds as well, and therefore (3) is true.

THEOREM 2. *Let the hypotheses on $f(x)$ be as in Theorem* 1, *and suppose that the surface of the unit sphere in B is strictly convex (i.e. that given a chord joining two points of the surface, all interior points of the chord are inside the sphere). Then* (2) *holds if and only if all the values of the function $f(x)$ are nonnegative multiples of a fixed nonzero vector in B.*

Proof: The sufficiency proof is so simple that we omit it.

Suppose that (2) holds; then (3) holds also, by Theorem 1. If $f(x)$ is identically zero there is nothing to prove. Hence suppose $y_0 = f(x_0) \neq 0$ for some x_0. Let $y = f(x)$. We have to express y as a nonnegative multiple of y_0. Dismissing the trivial case $y = 0$, let us write

$$u_0 = \frac{y_0}{\|y_0\|}, \qquad u = \frac{y}{\|y\|}.$$

Then $y_0+y=\|y_0\|u_0+\|y\|u$, and by (3)

$$1=\frac{\|y_0+y\|}{\|y_0\|+\|y\|}=\left\|\frac{\|y_0\|u_0+\|y\|u}{\|y_0\|+\|y\|}\right\|.$$

But then u_0,u, and

$$v=\frac{\|y_0\|u_0+\|y\|u}{\|y_0\|+\|y\|}$$

are all points on the surface of the unit sphere in B. However, if the points u_0, u are distinct, v is an interior point of the chord joining them. By the strict convexity of the surface of the unit sphere in B we conclude that u_0 and u coincide. In other words,

$$y=\frac{\|y\|}{\|y_0\|}y_0.$$

This completes the proof of the theorem.

The conclusion of Theorem 2 may be false if the surface of the unit sphere in B is not strictly convex. An example is afforded by taking B as the vector space of the complex numbers, with the norm defined by

$$\|\xi+i\eta\|=\max\{|\xi|,|\eta|\}.$$

Suppose $f(x)=\xi(x)+i\eta(x)$, where $\xi(x)\geqq 0$ and $|\eta(x)|\leqq\xi(x)$. Then it is easily seen that (2) holds; but the function values $f(x)$ need not be confined to a single ray through the origin.

Reference

1. E. Landau, Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie, Second Edition, Chelsea Publishing Co. New York, 1946, pp. 96–97.

A DIFFERENTIATION FORMULA

J. L. Gammel, Cornell University

1. Introduction. The question of changing the order of two infinite processes has in many cases been both interesting and fruitful. In tensor analysis, the form of important developments occasionally turns on the pattern of a commutation formula. In combinatorial analysis, commutation formulas have been used to discover relations among binomial coefficients.

This paper presents a generalization of a relationship which might be described as a commutation formula for the partial derivative of a total derivative of a point function. As an example of its application, the generalized relationship is used to prove a sum rule for binomial coefficients. Specifically, the theorem to be generalized is the following.

If $F(x^1, \cdots, x^N)$ is either a scalar or a vector point function of class M, then along a parameterized arc C: $x^a = x^a(t)$ of class M and for $0 \leqq P \leqq M$, $F^{(P)} = d^P F/dt^P$ is of functional order P,

$$F^{(P)} = F^{(P)}(x^1, \cdots, x^N, x^{(1)1}, \cdots, x^{(1)N}, \cdots, x^{(P)1}, \cdots, x^{(P)N}),$$

where $x^{(A)a} = d^A x^a/dt^A$. The following theorem,[1] in which

$$F^{(P)}_{;(A)a} = \partial F^{(P)}/\partial x^{(A)a},$$

$$\binom{P}{A} = \frac{P!}{A!(P-A)!},$$

and

$$F^{(P-A)}_a = d^{(P-A)}(\partial F/\partial x^a)/dt^{(P-A)},$$

is fundamental for the transformation theory associated with the extended coordinate transformation.

THEOREM: *If $F(x^1, \cdots, x^N)$ is either a scalar or a vector point function of class M, then along a parameterized arc C: $x^a = x^a(t)$ of class M and for $0 \leqq (P, A) \leqq M$*

$$F^{(P)}_{;(A)a} = \binom{P}{A} F^{(P-A)}_a.$$

Here, as in what follows, it is understood that

$$\binom{P}{A} = 0$$

when $A > P$ or $0 > A$, and that other "out of range" symbols are also zero. The purpose of the present note is to generalize this theorem to the case in which F is of functional order J.

2. The Formula. We shall establish the following result.

THEOREM: *If $F(x^1, \cdots, x^N, \cdots, x^{(J)1}, \cdots, x^{(J)N})$ is either a scalar or a vector point function of class M, then along a parameterized arc C: $x^a = x^a(t)$ of class $M+J$ and for $0 \leqq P \leqq M$, $0 \leqq A \leqq M+J$*

$$F^{(P)}_{;(A)a} = \sum_{i=0}^{J} \binom{P}{A-i} F^{(P+i-A)}_{;(i)a}.$$

Proof: For any J, the case $P = 0$, A arbitrary, and the cases $A \geqq P+J$, P arbitrary, are obvious. To complete the proof, it is sufficient to show that if the theorem is true for J, P, A then it is true for J, $P+1$, A. The proof follows.

[1] H. V. Craig, Vector and Tensor Analysis, p. 215, McGraw-Hill, 1943.

Differentiate $F^{(P)}$ with respect to t by the chain rule:

$$F^{(P+1)} = \sum_{B=0}^{P+J} F^{(P)}_{;(B)b} x^{(B+1)b},$$

and differentiate this with respect to $x^{(A)a}$:

$$F^{(P+1)}_{;(A)a} = \sum_{B=0}^{P+J} F^{(P)}_{;(B)b;(A)a} x^{(B+1)b} + \sum_{B=0}^{P+J} F^{(P)}_{;(B)b} \frac{\partial x^{(B+1)b}}{\partial x^{(A)a}}.$$

Now, because of the independence of the various quantities $x^{(A)a}$, the derivative $\partial x^{(B+1)b}/\partial x^{(A)a}$ which appears at the last of the foregoing equation may be replaced by the product of Kronecker deltas $\delta^{B+1}_{A}\delta^{b}_{a}$, and thus we may write

$$F^{(P+1)}_{;(A)a} = (F^{(P)}_{;(A)a})' + F^{(P)}_{;(A-1)a}.$$

If the formula is true for J, P, A then

$$F^{(P+1)}_{;(A)a} = \sum_{i=0}^{J}\left[\binom{P}{A-i} F^{(P+i-A+1)}_{;(i)a} + \binom{P}{A-i-1} F^{(P+i-A+1)}_{;(i)a}\right];$$

and since

$$\binom{P}{A-i} + \binom{P}{A-i-1} = \binom{P+1}{A-i},$$

we have

$$F^{(P+1)}_{;(A)a} = \sum_{i=0}^{J}\binom{P+1}{A-i} F^{(P+i-A+1)}_{(i)a}.$$

This is what was to have been shown.

As an example of the way in which the formula may be applied to discover relations among binomial coefficients, it will be shown that

$$\sum_{i=k}^{P}\binom{M}{B-i}\binom{P}{i-k} = \binom{P+M}{B-k},$$

where C is the smaller of B and $P+k$.

Proof: Now

$$F^{(P+M)}_{;(B)b} = [F^{(P)}]^{(M)}_{;(B)b}.$$

An immediate application of the generalized relationship gives

$$[F^{(P)}]^{(M)}_{;(B)b} = \sum_{i=0}^{P+J}\binom{M}{B-i}[F^{(P)}_{;(i)b}]^{(M+i-B)},$$

and applying it again to the quantity in the brackets we obtain

$$F_{;(B)b}^{(P+M)} = \sum_{i=0}^{P+J} \binom{M}{B-i} \sum_{k=0}^{J} \binom{P}{i-k} F_{(k)b}^{(M+i-B+P+k-i)}.$$

But applying the generalized relationship directly to the left-hand side we get

$$F_{;(B)b}^{(P+M)} = \sum_{k=0}^{J} \binom{P+M}{B-k} F_{(k)b}^{(P+M+k-B)}.$$

A comparison of the two results shows that

$$\sum_{i=0}^{P+J} \binom{M}{B-i}\binom{P}{i-k} = \binom{P+M}{B-k}.$$

But because of the second binomial coefficient on the left-hand side, i must be greater than k and less than $P+k$, which is less than $P+J$. Thus the lower limit for i can be replaced by k and the upper limit by $P+k$. Because of the first binomial coefficient on the left-hand side, the upper limit for i can be put equal to B if B is less than $P+k$. This completes the proof.

A PROOF OF INDEPENDENCE

HAO WANG, Cambridge, Massachusetts

Hilbert and Bernays give on p. 66 of their *Grundlagen der Mathematik*, vol. I (Berlin, 1934), a system of axioms for the propositional calculus. The system may be described as follows:

(1) *Any statement of one of the following form is a theorem*:

I.1. $A \to (B \to A)$. I.2. $(A \to (A \to B)) \to (A \to B)$. I.3. $(A \to B) \to ((B \to C) \to (A \to C))$.

II.1. $A \& B \to A$. II.2. $A \& B \to B$. II.3. $(A \to B) \to ((A \to C) \to (A \to B \& C))$.

III.1. $A \to A \vee B$. III.2. $B \to A \vee B$. III.3. $(A \to C) \to ((B \to C) \to (A \vee B \to C))$.

IV.1. $(A \sim B) \to (A \to B)$. IV.2. $(A \sim B) \to (B \to A)$. IV.3. $(A \to B) \to ((B \to A) \to (A \sim B))$.

V.1. $(A \to B) \to (\overline{B} \to \overline{A})$. V.2. $A \to \overline{\overline{A}}$. V.3. $\overline{\overline{A}} \to A$.

(2) *If A and $A \to B$ are theorems, then B is a theorem.*

They prove the independence of each of I.1—V.3 from the rest of the system on pp. 72–79, *ibid.*, and prove the independence of all of these except I.1 and I.3 with tables involving two or three truth values. The proofs of I.1 and I.3 make use of tables involving four truth values.

Now I propose to contribute an alternative proof for the independence of I.1 by a table which involves only three truth values. The table I use is such that when A and $A \to B$ both take the value 1, then B also takes the value 1. It is as follows.

A	B	$\overline{A}$	$A \vee B$	$A \& B$	$A \supset B$	$A \rightarrow B$
1	1	3	1	1	1	1
1	2	3	1	2	3	3
1	3	3	1	3	3	3
2	1	2	1	2	3	1
2	2	2	2	2	1	1
2	3	2	2	3	3	3
3	1	1	1	3	3	1
3	2	1	2	3	3	1
3	3	1	3	3	1	1

According to the table, all the axioms I.2–V.3 always take the value 1, while I.1, $(A \rightarrow (B \rightarrow A))$, takes the value 3 when A and B take the values 2 and 1 respectively.

It can be shown that no proof of independence involving only three truth values, whether with one or two designated truth values, can be constructed for I.3, and from this it follows that no proof can be constructed involving only two truth values. The proof consists essentially in constructing all possible truth tables for "$\rightarrow$", and observing that any truth table which preserves designated values in accordance with (2), and always assigns designated values to I.1 and I.2, also always assigns designated values to I.3. The number of tables which need be checked can be materially reduced by proper use of I.1 and I.2.

HYPERBOLIC FUNCTION SERIES ARISING FROM A SIMPLE POTENTIAL PROBLEM

J. G. MILLAR, Calgary Branch, University of Alberta

1. Introduction. If Laplace's equation in two dimensions,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0,$$

is solved by separating the variables, we obtain solutions of the type $\phi = \cos kx \cosh ky$, $\phi = \sinh kx \cos ky$, and so on. If the function, harmonic in a square region and constant on the boundaries, is expanded by means of these solutions, the identity (4) of this note is suggested, from which can be deduced many simple and interesting series involving the hyperbolic functions, which are not readily obtained by other methods.

2. The fundamental identity. Consider the function ϕ_1, harmonic inside the square $x = \pm \frac{1}{2}$, $y = \pm \frac{1}{2}$, with the boundary values

$$\phi_1 = 0 \quad \text{on} \quad x = \pm \tfrac{1}{2},$$
$$\phi_1 = 1 \quad \text{on} \quad y = \pm \tfrac{1}{2},$$

the values at the vertices being indeterminate. If such a function exists and can be expressed in terms of products of circular and hyperbolic functions, then

$$\phi_1 = \sum_{n=0}^{\infty} a_n \cos (2n+1)\pi x \cosh (2n+1)\pi y. \tag{1}$$

Sine and hyperbolic sine functions are ruled out by the fact that ϕ_1 must be an even function of x and y, while its vanishing at $x = \pm \frac{1}{2}$ fixes the factors involving x as $\cos (2n+1)\pi x$. Taking into account the second boundary condition, we must have, for $|x| < \frac{1}{2}$,

$$1 = \sum_{n=0}^{\infty} a_n \cos (2n+1)\pi x \cosh (2n+1)\frac{\pi}{2}.$$

Using the well-known Fourier expansion for the "square-wave" function,

$$1 = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n \cos (2n+1)\pi x}{2n+1}, \qquad |x| < \tfrac{1}{2}, \tag{2}$$

we must have, in (1),

$$a_n = \frac{4}{\pi} \frac{(-1)^n}{(2n+1) \cosh (2n+1)\frac{\pi}{2}}. \tag{3}$$

With this value of a_n in (1), we have a formal solution of our boundary-value problem.

We now consider the symmetrical problem with the boundary conditions

$$\phi_2 = 0 \quad \text{on} \quad y = \pm \tfrac{1}{2},$$
$$\phi_2 = 1 \quad \text{on} \quad x = \pm \tfrac{1}{2},$$

with a similar solution

$$\phi_2 = \sum_{n=0}^{\infty} a_n \cosh (2n+1)\pi x \cos (2n+1)\pi y,$$

where the value of a_n as given in (3) is unchanged.

Since the problem is linear, the two solutions may be superposed, giving the solution of the potential problem in which $\phi = 1$ on the sides of the square, except perhaps at the vertices. Since the unique solution of the case where $\phi = 1$ everywhere on the boundary is $\phi \equiv 1$, we are led to suspect strongly that $\phi_1 + \phi_2$ is identically unity inside the square; that is, that

$$\frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1) \cosh (2n+1)\frac{\pi}{2}} [\cos (2n+1)\pi x \cosh (2n+1)\pi y + \cosh (2n+1)\pi x \cos (2n+1)\pi y] = 1; \tag{4}$$

and this can in fact be shown to hold for all points of the square except the vertices.

3. Proof. The result (4) may be proved by the method above, but it would be necessary to invoke rather strong theorems of potential theory, and probably the simplest method is to use instead of (4) the series obtained by integrating it twice with respect to each variable, when difficulties due to lack of uniform convergence disappear. We shall need some further Fourier series. Series (2) is uniformly convergent in any range inside $-\frac{1}{2}<x<\frac{1}{2}$ excluding an arbitrarily small region about the end-points, and hence it can be integrated termwise in $-\frac{1}{2}\leqq x\leqq\frac{1}{2}$. Integrating (2) twice, first from 0 to x and then from $-\frac{1}{2}$ to x, we obtain

$$\frac{x^2}{2}-\frac{1}{8}=-\frac{4}{\pi^3}\sum_{n=0}^{\infty}\frac{(-1)^n\cos(2n+1)\pi x}{(2n+1)^3}. \tag{5}$$

Repeating the double integration, we have

$$\frac{x^4}{24}-\frac{x^2}{16}+\frac{5}{384}=\frac{4}{\pi^5}\sum_{n=0}^{\infty}\frac{(-1)^n\cos(2n+1)\pi x}{(2n+1)^5}. \tag{6}$$

Both series converge uniformly for $-\frac{1}{2}\leqq x\leqq\frac{1}{2}$.

Returning now to the conjectured identity (4), we integrate it twice with respect to each variable, the integration constants being supplied with the help of identities (5) and (6). The resulting equation is

$$\begin{aligned}-\frac{4}{\pi^5}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^5\cosh(2n+1)\frac{\pi}{2}}&[\cosh(2n+1)\pi x\cos(2n+1)\pi y\\&+\cos(2n+1)\pi x\cosh(2n+1)\pi y]\\=-\frac{1}{24}&\left(x^4-6x^2y^2+y^4+\frac{1}{4}\right).\end{aligned} \tag{7}$$

This result can be readily proved, since the series on the left-hand side, together with its second derivatives, is uniformly convergent in the square, and satisfies Laplace's equation, as does the polynomial on the right-hand side. From (6), both sides have the same values on the boundaries $x=\pm\frac{1}{2}$, $y=\pm\frac{1}{2}$. But for a closed region, the harmonic function with given boundary-values is unique, and thus (7) is an identity throughout the square. Equation (4) may now be proved by differentiating (7) twice with respect to each variable. All the series concerned are uniformly convergent in any closed region inside the square, and hence (4) is verified for all interior points of the square. From (2) it holds also on the boundaries, except at the vertices, where it is false.

4. Derived series. Identity (4) may be made the starting point for obtaining a large number of series involving $\cosh(2n+1)\pi/2$. Putting $x=y=0$, for example, we get

$$\frac{\pi}{8} = \frac{1}{\cosh\frac{\pi}{2}} - \frac{1}{3\cosh\frac{3\pi}{2}} + \frac{1}{5\cosh\frac{5\pi}{2}} - \cdots .$$

Other results are obtained by differentiating or integrating (4). Differentiating once with respect to x, and putting $y=0$, $\pi x=z$, we have

$$\frac{\sin z}{\cosh\frac{\pi}{2}} - \frac{\sin 3z}{\cosh\frac{3\pi}{2}} + \frac{\sin 5z}{\cosh\frac{5\pi}{2}} - \cdots$$
$$= \frac{\sinh z}{\cosh\frac{\pi}{2}} - \frac{\sinh 3z}{\cosh\frac{3\pi}{2}} + \frac{\sinh 5z}{\cosh\frac{5\pi}{2}} - \cdots ,$$

for $|z| < (\pi/2)$. Differentiating (4) twice with respect to each variable, obtaining as usual a series unformly convergent in any region inside the square, and putting $x=y=0$, we get

$$\frac{1^3}{\cosh\frac{\pi}{2}} - \frac{3^3}{\cosh\frac{3\pi}{2}} + \frac{5^3}{\cosh\frac{5\pi}{2}} - \cdots = 0.$$

Integrating (4) once with respect to each variable, and putting $x=y=\frac{1}{2}$, we have

$$\frac{\pi^3}{32} = \tanh\frac{\pi}{2} + \frac{1}{3^3}\tanh\frac{3\pi}{2} + \frac{1}{5^3}\tanh\frac{5\pi}{2} + \cdots .$$

The method here adopted may be extended to three or more dimensions. In three dimensions we arrive at such series as

$$\frac{\pi^2}{48} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^{m+n}}{(2m+1)(2n+1)\cosh\frac{\pi}{2}\sqrt{[(2m+1)^2+(2n+1)^2]}} .$$

These series can, of course, be obtained by other methods such as a contour integration or the transformation of the complete elliptic integral [1], but that given seems the simplest.

The fact that rectangular boundary problems could yield such identities was apparently first noted by Thomson (Lord Kelvin) and Tait [2] who saw that the expression they obtained for the torsional rigidity of a rectangular bar was unsymmetrical in the directions of the sides of the rectangle, and produced a non-trivial identity by equating the original expression to that obtained by symmetrical interchange.

References

1. Purser, F., Messenger of Math., vol. 11, 1882, p. 26.
2. Thomson and Tait, Treatise on Natural Philosophy, Camb. Univ. Press, 1903, vol. 2, p. 249.

CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, Haverford College and Massachusetts Institute of Technology

All material for this department should be sent to C. B. Allendoerfer, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

ON THE SYSTEM OF NATURAL NUMBERS

M. O. GONZALEZ, University of Havana and J. D. MANCILL, University of Alabama

1. Introduction. It seems highly desirable to be able to develop a simple and logical basis for the system of natural numbers, abstracted from one's own experiences, that would be suitable for presentation to students in College Algebra. This would have the effect of placing early emphasis on the logical structure of elementary algebra, similar to that in demonstrative geometry in secondary schools.

In this paper we shall establish explicitly and concisely the undefined elements, the defined elements, and the postulates which serve as a logical basis for the arithmetic of natural numbers. These fundamental concepts lead easily and naturally to the concept of natural number and the principle of mathematical induction, as well as the other well known properties of the system of natural numbers.

2. Undefined elements. We shall take the following concepts as undefined elements: *unity, set, belonging to a set, correspondence, and order.* The investigation of the origin of these concepts is a physiological and philosophical problem but not a mathematical one.

The notions of unity and plurality or set are acquired early in life, when one of the principal faculties of our mind, ability to discriminate, comes into action. We then begin to perceive distinctly the objects of the external world and we begin also to have some knowledge of our own individuality, thus originating those primary concepts of unity and plurality.

The entities or objects belonging to a set are called *elements* of the set. We shall denote the elements of the set by capital letters and the set itself by boldface capital letters. We shall express the fact that the set **M** has the elements A, B, C, D, $\cdots$, or that these elements belong to **M** by writing

$$\mathbf{M} = \mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}, \cdots .$$

For convenience, we shall introduce the concept of null-set (empty set), which has no elements. Since a set is here regarded as a collection or aggregate of things called elements, the null-set is, in general, excluded from consideration. Non-empty sets may be called *proper sets*. We shall consider only proper sets unless otherwise stated.

When we associate, by means of some well defined rule, each element of a set with one and only one element of another set, we say that a one to one correspondence has been established between the sets.

By order we mean, roughly speaking, some distribution of things in time, space, or according to a certain rule or criterion, and an ordered set according to one rule may not be ordered according to some other rule. We shall adopt the terms *preceding* and *succeeding* for describing rules of order. That is, for an ordered set these terms have precise meanings. Conversely, a set may be regarded as *ordered* when precise meaning has been attached to these terms, that is, when some criterion is stated to decide for two elements of the set which is the preceding one.

3. Assumptions. The descriptions of the elements unity, set, belonging to a set, correspondence, and order discussed in the preceding section are not to be considered as definitions of these concepts, but as descriptions of their general characteristics. In this section we shall state four postulates which these undefined elements are to satisfy. We shall see that these four assumptions are sufficient as fundamental postulates of the Arithmetic.

POSTULATE 1. *Given a set* **X** *there exists another set* **Y** *having no elements in common with* **X.**

This postulate establishes the existence of elements which do not belong to any set, transcending our faculty to find in nature or create in the imagination. It serves as the antecedent of Postulate 2 and jointly with it the base of the operation called addition of numbers to be defined later.

POSTULATE 2. *Given a set* **X,** *it is possible to form another set either by removing an element from* **X** *or by adjoining another element to those of* **X.**

POSTULATE 3. *Given the sets* **X** *and* **Y,** *it is possible to choose an element in each as corresponding elements.*

This postulate is known as the principle of free choice or axiom of Zermelo.

POSTULATE 4. *For any elements* A, B, *and* C *of an ordered set we have*

(1) A *does not precede* A,
(2) *If* A *precedes* B, *then* B *does not precede* A,
(3) *If* A *precedes* B *and* B *precedes* C, *then* A *precedes* C.

These conditions characterize what is more specifically known as *linear order.*

4. Fundamental definitions. We are now ready to define certain concepts explicitly.

DEFINITION 1. *A set is said to be determined or defined when there exists some rule to decide whether a given object belongs or does not belong to the set.*

DEFINITION 2. *The sets* **M** *and* **N** *are said to be equal whenever every element of* **M** *belongs to* **N** *and conversely.*

The equality of the sets **M** and **N** is denoted by the symbolism

$$\mathbf{M} = \mathbf{N}.$$

DEFINITION 3. *The set* **H** *is said to be a proper subset of the set* **K** *if the elements of* **H** *belong to* **K** *but there are elements in* **K** *which do not belong to* **H.**

We shall indicate that **H** is a subset of **K** by the notation

$$\mathbf{H} < \mathbf{K} \quad \text{or} \quad \mathbf{K} > \mathbf{H}.$$

DEFINITION 4. *If in an ordered set there exists an element preceding all the remaining elements, this element is called the first element of the set. If there exists an element which does not precede any element it is called the last element of the set.*

DEFINITION 5. *Elements* A *and* B *of an ordered set are said to be consecutive when there does not exist an element* C *such that* A *precedes* C *and* C *precedes* B. *In this case the element* A *is said also to be the immediate predecessor of* B.

Sets in mathematics may be finite, as the set composed of the sides of a polygon, or infinite, as the set of points in a circle. The usual procedure in the past has been to define infinite sets first and then define a finite set as one which is not infinite. It seems far more natural and, therefore, desirable to define finite sets first. That it is possible to define finite sets in terms of the notion of order is shown by the next definition.

DEFINITION 6. *A set* **M** *is said to be finite if it can be ordered in such a way as to make every subset of* **M** *have a first and a last element.*

The following properties are immediate consequences of this definition:

(1) *All finite sets have a first and a last element.*

(2) *All subsets of a finite set are finite.*

(3) *The elements of a finite set are consecutive.*

It is always assumed that the same ordering is retained in the subsets of an ordered set.

DEFINITION 7. *A set that is not finite is said to be infinite.*

DEFINITION 8. *A simply infinite set* **S** *is an infinite set such that*

(1) *Every subset of* **S** *has a first element,*

(2) *Every element of* **S** *except the first has an immediate predecessor.*

It follows from the first condition in this definition that every element of **S** has an immediate successor. The elements of a simply infinite set are, therefore, consecutive. The simply infinite sets differ from the finite sets in the fact that they do not have a last element.

As an example, let us consider the set of points A, A′, A″, A‴, · · · , B, where A′ is the mid-point of the segment AB; A″ is the mid-point of A′B; A‴ is the mid-point of A″B; and so forth.

This set may be ordered by letting the word *precede* mean *to the left of*, and then the first element is A and the last element is B. The set is infinite, although it has a first and a last element. This shows why we stated the condition for a

finite set in terms of its subsets, requiring for each one the property of having a first and a last element. The converse of this is not true, that is, if the set has a first and a last element it does not follow that all subsets have a first and a last element. Thus, in the above example, the subset A, A′, A″, · · · does not have a last element.

The total set in the above example is not simply infinite, since the second condition of Definition 8 is not satisfied because the element B does not have an immediate predecessor. The set obtained by removing the element B is simply infinite, since it has the properties required in Definition 8.

DEFINITION 9. *Two sets are said to be equivalent when there exists a one to one correspondence between their elements.*

It follows easily that the relation of equivalence is *reflexive*, *symmetric*, and *transitive*.

We shall indicate that two sets **X** and **Y** are equivalent by the notation

$$\mathbf{X} \cdot\cdot \mathbf{Y}.$$

DEFINITION 10. *When two sets are equivalent we say they have the same number of elements.*

Thus the *concept of number* results from considering equivalent sets when abstraction is made of all other characteristics of their elements such as kind, color, size, etc. A similar mental process leads to the formation of all our general or abstract concepts. Thus, for example, by looking at different flowers and neglecting the particular characteristics of each one, but retaining their essential common properties (those concerning the stamen and pistil), we arrive at the general concept of flower.

In analogous fashion the number concept is an abstraction elaborated spontaneously in our mind for the purpose of representing equivalent sets.

Definitions of this type are called *definitions by abstraction*. It is the most suitable manner of definition for those concepts which are formed by considering in some sense the equality or equivalence of objects. Two objects are said to be equal or equivalent when they have in common a certain specified property, being capable of mutual substitution with respect to this property. From each kind of equivalence a new concept is created.

5. Principle of Complete Induction. *If the first element of a finite or simply infinite set* **X** *has a certain property, and if from the assumption that some element has the property it follows that the next element also has that property, then all the elements of the set* **X** *have said property.*

Suppose on the contrary that some elements of **X** do not have said property and let **Z** denote the set of such elements. The set **Z,** being a subset of **X,** has a first element K. That is, K is the first element not having the said property. Let J denote the immediate predecessor of K. Since J has the said property, it follows

that K has the property, which contradicts the definition of K. Therefore, the principle is proved.

THEOREM. *A finite set can be made empty by removing its elements one at a time.*

Let the given finite set be

$$\mathbf{X} = \mathrm{A}, \mathrm{B}, \mathrm{C}, \cdots, \mathrm{H}, \mathrm{I}, \cdots, \mathrm{L}.$$

The theorem is true for the set

$$\mathbf{X}' = \mathrm{A}$$

since the single element A can be removed by Postulate 2. If the theorem is true for the set

$$\mathbf{X}'' = \mathrm{A}, \mathrm{B}, \mathrm{C}, \cdots, \mathrm{H},$$

it is true for the set

$$\mathbf{X}''' = \mathrm{A}, \mathrm{B}, \mathrm{C}, \cdots, \mathrm{H}, \mathrm{I}$$

by removing the element I and thus obtaining the set $\mathbf{X}''$ having the property. By the principle of induction the property is certain for the entire set.

6. Properties of Equivalent Sets. We shall now state certain useful properties of equivalent sets.

(1) *If the sets* **X** *and* **Y** *are equivalent, then the sets resulting from removing an element from each set are equivalent; and the sets resulting from adjoining an element to each set are equivalent.*

(2) *Finite sets* **X** *and* **Y** *are either equivalent or one of them is equivalent to a part of the other.*

This follows easily from the theorem on the exhaustion of finite sets.

(3) THEOREM OF DIRICHLET. *No finite set is equivalent to a part of itself.*

An indirect proof can be easily made, based on the theorem on exhaustion of finite sets.

This property is not true of infinite sets. In fact, Dedekind and others have used this property to distinguish finite and infinite sets. That is, they defined an infinite set as one equivalent to a part of itself and then defined finite set as one that is not infinite. The approach used here, that of defining finite sets first, seems far more natural and it constitutes a direct approach to the most important fundamental concepts of arithmetic.

(4) *If the finite sets* **X** *and* **Y** *are equivalent by a certain mode of correspondence, then they are equivalent under any mode of correspondence.*

Let us suppose that

$$\mathbf{X} \cdot \cdot \mathbf{Y}. \tag{1}$$

If the correspondence is established in some other manner and the results are not equivalent, then we have by Property 2

(2) $$\mathbf{X}' \cdot\cdot \mathbf{Y} \quad \text{or} \quad \mathbf{X} \cdot\cdot \mathbf{Y}'$$

where $\mathbf{X}'$ is a part of $\mathbf{X}$ and $\mathbf{Y}'$ a part of $\mathbf{Y}$. But it follows from (1) and (2), and the transitive property of equivalence that

$$\mathbf{X}' \cdot\cdot \mathbf{X} \quad \text{or} \quad \mathbf{Y}' \cdot\cdot \mathbf{Y}$$

which contradicts the Theorem of Dirichlet.

COROLLARY. *If the finite sets* $\mathbf{X}$ *and* $\mathbf{Y}$ *are not equivalent under a certain mode of correspondence, then they remain non-equivalent under any mode of correspondence.*

(5) *Every set equivalent to a finite set is also finite.*

A finite set has the property, by definition, that every subset has a first and a last element. It is obvious that any set equivalent to a finite set would have the same property in view of the correspondence between the sets.

(6) *All orderings of a finite set are equivalent.*

For, any finite set is equivalent to itself and, therefore, equivalent to any ordering of it by Property 4.

COROLLARY. *If a set is infinite with a certain ordering, then it remains infinite under any ordering.*

7. Fundamental Succession. The above properties of equivalent sets, which are intuitively known and applied by everyone, lead to the construction of a sequence of sets such that no two of them are equivalent and such that one of them is equivalent to an arbitrary given finite set. For, let A be an object; by adjoining an object B, we obtain a set A, B. By adjoining another object C, we obtain a set A, B, C. Proceeding always in the same manner (Postulate 2) we form the so-called fundamental sequence of sets:

$$\text{A}; \quad \text{A, B}; \quad \text{A, B, C}; \quad \text{A, B, C, D}; \cdots .$$

In this sequence each set is a subset of the succeeding sets, whence no two of them are equivalent. Also, it is easily seen that any finite set $\mathbf{W}$ is equivalent to one of these sets in the fundamental sequence. For if $\mathbf{W}$ is equivalent to the sets $\mathbf{X}$ and $\mathbf{Y}$ of the fundamental sequence, then $\mathbf{X}$ and $\mathbf{Y}$ are equivalent which we have seen is not true.

Since every finite set is equivalent to one and only one set of the fundamental sequence, the human mind has created number symbols to represent the sets which are equivalent to each one in the sequence. These number symbols have names. Thus, the number corresponding to the sets equivalent to (A) is denoted by the symbol 1 and is called *one;* the number corresponding to the sets which are equivalent to (A, B) is denoted by 2 and is called *two;* the number corresponding to the sets which are equivalent to (A, B, C) is denoted by the symbol 3 and is called *three;* and so forth.

The concept of these particular numbers develops little by little in the child's mind and the general concept of number is formed with them, that is,

the concept of the numbers as the common entity to any equivalent sets (Def. 10). This general concept of number may be thought of as a *genus* for which the particular numbers 1, 2, 3, $\cdots$ are the *species*.

Since the fundamental sequence constitutes an infinite set of sets (simply infinite) it is impracticable to have distinct and unrelated symbols for each number. This difficulty has been overcome by establishing a set of rules by means of which all numbers can be represented with a finite number of symbols. This set of rules is called a system of numeration. The system in use, the so-called *decimal system*, makes use of the symbols 1, 2, 3, 4, 5, 6, 7, 8, 9, together with the symbol 0 (zero) which is taken to represent the null or empty set.

The set of numbers

$$1, 2, 3, \cdots, 9, 10, 11, \cdots, 101, 102, 103, \cdots$$

which represents the fundamental sequence of sets in the manner described above is called the *sequence of natural numbers*.

8. Cardinal Numbers. Take a set of objects of any kind, for example

$$\mathbf{X} = \mathbf{M},\ \mathbf{N},\ \mathbf{P},\ \mathbf{S},\ \mathbf{T}.$$

If we suppose for a moment that this set contains only the element M, it is equivalent to (A) and is represented by the number 1, which we shall indicate by writing M_1. Similarly, we write M_1, N_2 if the set contains only the elements M and N. Continuing this way, we write

$$M_1,\ N_2,\ P_3,\ S_4,\ T_5$$

to indicate that the set **X**, considered as having all its elements, is equivalent to (A, B, C, D, E) of the fundamental sequence and represented by the number 5. It is seen that the number of a set is assigned to the last element and for that reason is called *cardinal* (principal) *number* of the set. In practice, this method of designating the elements in an ordering of a set is called counting the elements of the set. The number corresponding to the last element is the cardinal number of the set.

Observing that if we count the elements in distinct orderings we obtain the same cardinal number, because we have already seen that all orderings of a finite set are equivalent, we have

The cardinal number of a set is independent of the ordering of the set.

This is called, according to Schröder, the principle of the invariance of the number.

The sets of the fundamental succession are considered ordered in the following way: A set precedes another if it is a part of the other, as for example, the set (A, B, C) precedes the set (A, B, C, D, E). The set of natural numbers becomes ordered by correspondence with the fundamental succession, that is, for example, 3 precedes 5. Finally, the sets in nature may be distinctly ordered by correspondence, by means of the process called enumeration, considering that an

object precedes another when its attached number precedes that of the other. Numbers used for this purpose are called *ordinal* numbers.

9. Equality and Inequality of Natural Numbers. Let **M** and **N** denote any two finite sets with cardinal numbers m and n respectively. It follows from Property 2 of Section 6 that one and only one of the following cases can arise:

(a) $\mathbf{M} \cdot\cdot \mathbf{N}$, in which case m and n represent the same number and this is denoted by the symbolism $m = n$;

(b) $\mathbf{M} \cdot\cdot \mathbf{N}'$, where $\mathbf{N}'$ is a part of **N**, in which case m is said to be less than n and this is denoted by the symbolism $m < n$;

(c) $\mathbf{M}' \cdot\cdot \mathbf{N}$, where $\mathbf{M}'$ is a part of **M**, in which case m is said to be greater than n and this is denoted by the symbolism $m > n$.

It is easily shown that the relation of equality of numbers is *reflexive*, *symmetric*, and *transitive* and that the inequality relations $>$ and $<$ are not reflexive nor symmetric but are transitive.

10. Sum of Natural Numbers. We shall first consider the sum of finite sets.

EXISTENCE THEOREM. *Given the finite sets* $\mathbf{A}, \mathbf{B}, \mathbf{C}, \cdots, \mathbf{N}$ *which have no elements in common, then there exists a set* **S** *formed exclusively of the elements of the sets. The set* **S** *is called the sum of the sets* $\mathbf{A}, \mathbf{B}, \mathbf{C}, \cdots, \mathbf{N}$.

This follows from the principle of exhaustion of finite sets and Postulate 2.

THEOREM. *The sum of a finite number of finite sets is a finite set.*

For, the two conditions of Definition 6 are satisfied:

(1) The sum may be ordered. It will suffice to order the set of sets and the elements of each set. This is possible since all sets involved are finite. Once this has been done, for any two elements P and Q we have the following: If they are in different sets, we shall consider P as preceding Q when P belongs to a set which precedes the set to which Q belongs; if they belong to the same set, we keep for them the same order as they have in the set.

(2) Every subset of the total set has a first and a last element. For, among the sets there will be a first one and a last one containing elements of the chosen subset. These elements will form within each set partial subsets with a first and a last element.

The following laws of sums of finite sets are easily proved:

(1) UNIFORM: *The sum of equivalent sets results in equivalent sets.*

(2) COMMUTATIVE: *The sum set* **S** *of the sets* $\mathbf{A}, \mathbf{B}, \mathbf{C}, \cdots, \mathbf{N}$ *is independent of the order in which they are added.*

(3) ASSOCIATIVE: *The same sum set* **S** *is obtained by adding any two of the sets* $\mathbf{A}, \mathbf{B}, \mathbf{C}, \cdots, \mathbf{N}$ *and to that result the remaining sets.*

(4) MONOTONIC: *If* $\mathbf{A} > \mathbf{B}$, *then* $(\mathbf{A}, \mathbf{C}) > (\mathbf{B}, \mathbf{C})$ *for every finite set* **C**.

To define the *sum of natural numbers*, suppose that the finite sets $\mathbf{A}, \mathbf{B}, \mathbf{C}, \cdots, \mathbf{N}$ have the cardinal numbers $a, b, c, \cdots, n$, respectively. Then the sum set **S** is finite and its cardinal number s is defined to be the sum of the numbers $a, b, c, \cdots, n$, and is written

$$s = a + b + c + \cdots + n.$$

The operation of addition satisfies the following laws which are entirely analogous to the above laws for sum of sets.

(1) UNIFORM: *If* $a=a', b=b', \cdots, n=n'$, *then* $a+b+\cdots+n=a'+b'+\cdots+n'$.

(2) COMMUTATIVE: $a+b=b+a$.

(3) ASSOCIATIVE: $a+b+c=(a+b)+c=a+(b+c)$.

(4) MONOTONIC: *If* $a>b$, *then* $a+c>b+c$ *for every number* c.

These laws follow from the corresponding laws for sums of sets.

A number which if added to any number x results in the same number x is called the *unit of addition*. It is easily seen that the unit of addition is zero, that is,

$$a + 0 = a.$$

Furthermore, from the monotonic law, if $c>0$, then

$$a + c > a.$$

11. Multiplication of Natural Numbers. We call the *product* of a number a and a number b $(b>1)$ the sum of b summands each equal to a, and is written ab or $a\times b$.

The operation of multiplication obeys the following laws:

(1) UNIFORM: *If* $a=a'$ *and* $b=b'$, *then* $ab=a'b'$.

(2) DISTRIBUTIVE: $(a+b+c)n=an+bn+cn$.

(3) COMMUTATIVE: $ab=ba$.

(4) ASSOCIATIVE: $abc=(ab)c=a(bc)$.

(5) MONOTONIC: *If* $a>b$, *then* $ac>bc$, *for* $c>0$.

The products $1\times b$ and $0\times b$ are expressed by

$$1 \times b = 1 + 1 + \cdots + 1 = b,$$
$$0 \times b = 0 + 0 + \cdots + 0 = 0,$$

according to the above definition. This definition has no meaning, however, in the case of the expressions $b\times 1$ and $b\times 0$. With the object of preserving the commutative law for multiplication of any two numbers without exception, we shall agree to attribute to the expressions $b\times 1$ and $b\times 0$ the same values as to $1\times b$ and $0\times b$, respectively. That is, we agree upon the following relations:

$$b \times 1 = 1 \times b = b, \qquad b \times 0 = 0 \times b = 0,$$

and in particular $1\times 1=1$, $1\times 0=0$, and $0\times 0=0$. It is easily verified that the above five laws of multiplication are valid under these interpretations.

The operations of subtraction and division can now be defined and the arithmetic of natural numbers further developed. This development will not be carried further here since our primary object has been to develop a simple and logical basis for the arithmetic of natural numbers, suitable for presentation to students in what might be called College Arithmetic.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 901. *Proposed by Leo Moser, University of Manitoba*

A company offers its 350 employees a bonus of \$10.00 to each male and \$8.25 to each female. All the females accept, but a certain percentage of the males refuse to accept. Knowing what this percentage is, one can deduce the total amount paid out. What is the total amount paid to the women?

E 902. *Proposed by R. V. Andree's class in Solid Analytics, University of Oklahoma*

A similar argument to that used in E 832 [1949, 407] may be used to "prove" the following theorem: "If a sphere has a center with at least one irrational coordinate, then there are at most three points on the sphere with rational coordinates." Give a counter example to show that this theorem is not true and discuss the point at which the "proof" breaks down.

E 903. *Proposed by F. A. E. Pirani, Carnegie Institute of Technology*

Show that every even integer greater than 46 can be expressed as the sum of two abundant numbers. (See E 848, [1949, 478].)

E 904. *Proposed by C. S. Ogilvy, Columbia University*

Find the equation of the locus of centers of curvature of $y=x^n$ at the point (1, 1) as n varies continuously from $-\infty$ to $+\infty$.

E 905. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let A_1 and A_2, B_1 and B_2, C_1 and C_2 be the points of intersection with the sides BC, CA, AB of a triangle ABC of the polars of a point P with respect to the escribed circles (I_b) and (I_c), (I_c) and (I_a), (I_a) and (I_b). For what position of P will the points A_1 and A_2, B_1 and B_2, C_1 and C_2 be, simultaneously, symmetric with respect to the midpoints of BC, CA, AB?

SOLUTIONS

A Hyperboloid of One Sheet

E 868 [1949, 338]. *Proposed by P. D. Thomas, Washington, D. C.*

Let P and Q be, respectively, the feet of the common perpendicular to two

fixed skew lines p and q. A variable line r meets p in R and q in S. Find the locus of r if the volume of the tetrahedron $PQRS$ is constant. Also find the locus of the centroid of $PQRS$.

Solution by N. D. Lane, Acadia University, Nova Scotia. Let P be the origin, p the x-axis, PQ the z-axis, Q the point $(0, 0, k)$, q the line $y=mx$, $z=k$, R the point $(a, 0, 0)$, and S the point (b, mb, k). Then equations for r are

$$(x - a)/(b - a) = y/mb = z/k,$$

from which

$$a = k(y - mx)/m(z - k) \quad \text{and} \quad b = yk/zm.$$

Now the constant volume, V, of the tetrahedron $PQRS$ is

$$V = \pm (1/6)\begin{vmatrix} a & 0 & 0 & 1 \\ b & mb & k & 1 \\ 0 & 0 & k & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \pm abmk/6,$$

whence

$$ab = \pm 6V/mk = k^2y(y - mx)/m^2z(z - k),$$

and the equation satisfied by any point on r is

$$k^3y^2 \mp 6Vmz^2 - k^3mxy \pm 6mVkz = 0,$$

which represents a hyperboloid of one sheet.

For the centroid $(\bar{x}, \bar{y}, \bar{z})$ we have

$$\bar{x} = (a + b)/4, \qquad \bar{y} = mb/4, \qquad \bar{z} = 2k/4,$$

whence

$$a = 4\bar{x} - b = 4\bar{x} - 4\bar{y}/m, \qquad b = 4\bar{y}/m,$$

and the relation $ab = \pm 6V/mk$ becomes

$$8k\bar{y}^2 - 8km\bar{x}\bar{y} \pm 3mV = 0.$$

Thus the locus of the centroid is the above hyperbola in the plane $z=k/2$.

Also solved by J. E. Darraugh, Roger Lessard, and the proposer.

A Polynomial the Square of Another Polynomial

E 869 [1949, 338]. *Proposed by P. T. Bateman, Institute for Advanced Study*

If a polynomial $f(x)$ with integral coefficients has the property that $f(n)$ is a perfect square for all integers n, then $f(x)$ is the square of another polynomial with integral coefficients.

Solution by W. H. J. Fuchs, Cornell University. The given problem is the special case $g(x)=x^2$ of the following more general

THEOREM: *If $f(x)$ and $g(x)$ are polynomials and if for every integer $p>p_0$ there is an integer $q=g(p)>0$ such that $f(p)=g(q)$, then $f(x)=g(h(x))$, where h is a polynomial. If f and g have integral coefficients and g has leading coefficient* 1, *then h also has integral coefficients.*

Proof: We note that $g(y)=f(x)$ defines y as an algebraic function of x. If g is of degree n, f of degree m, then for large x

$$y = x^{m/n}\sum_{k=0}^{\infty}\gamma_k x^{-k/n}. \tag{1}$$

If x is a positive integer, then, for a suitable determination of the nth root of x, y is also a positive integer. This determination is the same for all large x, since $y=\gamma_q x^{(m-q)/n}(1+o(1))$, where q is the least integer with $\gamma_q\neq 0$. There is no loss of generality in assuming that $x^{1/n}$ has to be chosen positive.

Let $x=v^n p$, $(v=1, 2, \cdots, h)$. By (1)

$$y = q(v^n p) = \sum_{k=0}^{h-1}\gamma_k p^{(m-k)/n}v^{m-k} + O(p^{(m-k)/n}) \tag{2}$$

as $p\to\infty$. Regard (2) as a system of equations for the unknowns $\gamma_k p^{(m-k)/n}$, $(k=0, 1, \cdots, h-1)$. The determinant of this system is

$$\Delta = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 2^m & 2^{m-1} & \cdots & 2^{m-h+1} \\ \cdot & \cdot & \cdots & \cdot \\ h^m & h^{m-1} & \cdots & h^{m-h+1} \end{vmatrix} \neq 0.$$

Solving (2),

$$\Delta\gamma_k p^{(m-k)/n} = \text{integer} + O(p^{(m-k)/n}), \qquad (k = 0, 1, \cdots, h-1). \tag{3}$$

If $h>k>m$ this is impossible unless $\gamma_k=0$. Hence (1) contains no negative powers of $x^{1/n}$ and so the O-term in (2), and therefore in (3), does not occur if $h=m+1$. Then (3) shows that $\gamma_k p^{(m-k)/n}$ is rational for every large p. This is possible only if $\gamma_k=0$ for all $k\not\equiv m \pmod n$. Hence (1) reduces to a polynomial $h(x)$ in x. By (3) $h(x)$ has rational coefficients.

To prove the last part of the theorem observe that if $N>1$ is the least common denominator of the coefficients of $h(x)$, then N^p is the least common denominator of the coefficients of $h^p(x)$, by a well known theorem. Hence $g(h(x))$ cannot have integral coefficients, since $N^{n-1}g(h(x))$ does not have integral coefficients ($n=$degree of g).

Also solved by L. M. Kelly.

Alex Rosenberg said that the problem, with a corresponding generalization to higher powers, seems first to have been proposed by Ch. Brisse as problem

No. 37 on p. 10 of vol. 1 of the *Intermédiaire des Mathématiciens* (1894). Two correct solutions were published in the same journal by I. Ivanoff (p. 74, vol. 1, 1894) and J. Franel (p. 94, vol. 2, 1895). The problem was then again proposed by R. Jentzsch in the *Archiv der Mathematik und Physik* (ser. 3, vol. 19, p. 361, 1912), and a third distinct solution was published in the same journal by W. Grosch (ser. 3, vol. 21, p. 368, 1913). The first two solutions are elementary in character, whereas the third solution is somewhat more sophisticated.

Also see problem 114, p. 135 of Pólya and Szegö, *Aufgaben und Lehrsätze aus der Analysis*, II.

Forty-five Degree Parallelograms

E 870 [1949, 338]. *Proposed by Joseph Rosenbaum, Hartford, Conn.*

Characterize quadrilaterals $A_1B_1C_1D_1$ such that if A_2, B_2, C_2, D_2 are the circumcenters of $A_1B_1C_1$, $B_1C_1D_1$, $C_1D_1A_1$, $D_1A_1B_1$, then A_1, B_1, C_1, D_1 are the circumcenters of $A_2B_2C_2$, $B_2C_2D_2$, $C_2D_2A_2$, $D_2A_2B_2$.

Solution by S. T. Thompson, Tacoma, Washington. By hypothesis

$$A_1A_2 = A_2C_1 = C_1C_2 = C_2A_1 = A_1B_2 = A_2B_1 = C_1D_2 = C_2D_1, \tag{1}$$

$$B_1B_2 = B_2D_1 = D_1D_2 = D_2B_1 = B_1C_2 = B_2C_1 = D_1A_2 = D_2A_1. \tag{2}$$

From the first three equalities of (1) and the first three of (2) it follows that $A_1A_2C_1C_2$ and $B_1B_2D_1D_2$ are rhombi. From the last three equalities of (1) and the last three of (2) it follows that the two rhombi are squares with a common center. This, in turn, guarantees that $A_1B_1C_1D_1$ is a parallelogram. Moreover, since $A_2A_1=A_2B_1=A_2C_1$, we have that angle $A_1B_1C_1$ equals 45° or 135°. Thus $A_1B_1C_1D_1$ is a 45° parallelogram. One can also easily show, conversely, that a 45° parallelogram satisfies the required conditions.

If the words "taken in some order" should be tacked onto the statement of the problem, then we would have, in addition to 45° parallelograms, all orthocentric quadruples.

Also solved by R. G. Buschman, L. M. Kelly, and C. S. Ogilvy. The proposer stated that the corresponding problem for five points in space seems to be not easy.

Values of a Parameter in a Trigonometric Identity

E 871 [1949, 403]. *Proposed by W. R. Ransom, Tufts College*

Is there any value of k other than $k=1$ for which $k \tan x \cot kx = 1$?

Solution by N. D. Lane, St. Andrew's College, Ontario. If $k=0$ the left side is indeterminate. Consider $y_1=k \tan x$ and $y_2=\tan kx$. If $y_1=y_2$ in any interval $a \leqq x \leqq b$ for which y_1 is finite, then $dy_1/dx=dy_2/dx$ in that interval. That is, $k \sec^2 x = k \sec^2 kx$, and we must have $k=\pm 1$. Clearly each of these values is acceptable.

Also solved by P. M. Anselone, W. D. Berg, A. B. Boggs, N. G. Gunderson,

Vern Hoggatt, M. S. Klamkin, Roger Lessard, C. S. Ogilvy, L. B. Rall, A. P. Rhodes, L. A. Ringenberg, and J. W. Sawyer.

The Enthusiastic Problemist

E 872 [1949, 403]. *Proposed by Leo Moser, University of Manitoba*

An enthusiastic problemist proposes at least one problem every day. In order not to overwork problem editors, however, he does not propose more than 730 problems a year. Given any positive integer n, show that he proposes exactly n problems in some set of consecutive whole days. (Dedicated to Victor Thébault.)

Solution by the Proposer. Let S_i denote the number of problems proposed in the first i days, starting today, say. Consider the set of integers $\{S_i\}$, $i=1, 2, \cdots, N$, where $N=2921n$. Notice that $2921n$ days $\leqq 8n$ years, taking leap years into account. Consider also the set of integers $\{n+S_i\}$, $i=1, 2, \cdots, N$. In the two sets together we have $2N=5842n$ integers, the largest of which is $n+S_N \leqq n+8n(730)=5841n$. Hence there is at least one integer common to both sets, say $S_j=S_k+n$, and exactly n problems will be proposed on the days numbered $k+1, k+2, \cdots, j$.

A Generalized Scalar Product

E 873 [1949, 404]. *Proposed by I. M. Hostetter, Oregon State College*

Let a vector be interpreted as a directed line segment and let $A * B$ be a symbolic product of two arbitrary vectors A and B such that (1) $A * B$ is a scalar, (2) $A * B = B * A$, (3) $A * (B+C) = A * B + A * C$. Is there an interpretation of $A * B$ other than the familiar inner, or scalar, product: $|A|\ |B| \cos\theta$, where $|A|$, $|B|$ are the (positive) lengths of A and B, and θ is the angle between A and B?

Solution by C. M. Ablow, Brown University. Under the very weak assumption that $f(x)$ is bounded in the neighborhood of some point, one may prove the

Lemma: If, for all x and y, $f(x+y)=f(x)+f(y)$, then $f(x)=kx$, where k is a constant.

Returning to the problem, let the components in rectangular coordinates of vector A be indicated by (a_1, a_2, a_3). Consider vectors A, B, C with but one nonzero component each, a_i, b_j, c_j, (note that B and C have the same component appearing), and denote these vectors by (a_i), (b_j), (c_j) respectively. Then (3) becomes

$$(a_i) * (b_j + c_j) = (a_i) * (b_j) + (a_i) * (c_j).$$

Thinking of a_i as fixed and b_j and c_j arbitrary this last is of the form of the lemma. Hence $(a_i) * (b_j) = kb_j$, where k is a number independent of b_j. However, (2) shows that we must have $(a_i) * (b_j) = k_{ij}a_ib_j$, where k_{ij} is now a constant, being independent of both variables, for which $k_{ij}=k_{ji}$. Now

$$(a_i) * (b_1, b_2, b_3) = (a_i) * (b_1) + (a_i) * (b_2) + (a_i) * (b_3)$$

$$= k_{i1}a_ib_1 + k_{i2}a_ib_2 + k_{i3}a_ib_3,$$

and applying this last to $(a_1, a_2, a_3) * (b_1, b_2, b_3)$ we see finally that our most general result is

$$A * B = \sum_{i,j=1}^{3} k_{ij}a_ib_j,$$

with $k_{ij}=k_{ji}$ constants.

Also solved by Vern Hoggatt and M. S. Klamkin who gave the rather trivial interpretations $A * B = -A \cdot B$ and $A * B = kA \cdot B$ respectively.

Editorial Note. The proposer is primarily interested in a geometrical or physical interpretation. Can a nice geometrical meaning be assigned to the above general symmetric bilinear form?

The Life-Buoy

E 875 [1949, 404]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

A life-buoy is formed by two right circular cones joined by their common base. (1) Show that the centroid of the solid coincides with the centroid of the vertices of the quadrangle obtained by cutting the surface of the buoy by an axial plane. (2) Determine the ratio of the altitudes of the two cones in order that the buoy may be in equilibrium when it rests, on a horizontal plane, on one of the two surfaces, or indifferently on one or the other of the two surfaces, of the two cones.

Solution by Vern Hoggatt, Oregon State College. Take the x-axis coincident with the axis of the buoy, the origin at the center of the common base of the cones. Let the altitudes of the cones be h_1 and h_2, $(h_2 \geqq h_1)$, and the radius of the common base be r. Since the centroid of a right circular cone lies on the axis of the cone at one fourth of the altitude from the base plane we have, where $\bar{x}$ is the x-coördinate of the centroid of the buoy,

$$(1/3)\pi r^2 h_2(h_2/4 - \bar{x}) = (1/3)\pi r^2 h_1(h_1/4 + \bar{x}),$$

or $\bar{x}=(h_2-h_1)/4$. This is also the x-coördinate of the centroid of the vertices of the quadrangle. The buoy will rest indifferently on either conical surface if and only if the perpendicular dropped from the centroid of the buoy upon an element of the smaller cone will hit that element. This happens if and only if $\bar{x}/r \leqq r/h_1$, or if and only if

$$h_2 \leqq (4r^2 + h_1^2)/h_1.$$

If h_2 exceeds this value the buoy will always come to rest on the surface of the larger cone.

Also solved by D. H. Browne, N. D. Lane, W. O. Pennell, and the proposer.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results found in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4380. *Proposed by P. A. Piza, San Juan, Puerto Rico*

For arbitrary positive integers n and k let

$$S_1(n, k) = 1^k + 2^k + \cdots + n^k,$$

and put

$$S_{p+1}(n, k) = S_p(1, k) + S_p(2, k) + \cdots + S_p(n, k)$$

with $p = 1, 2, \cdots$. Thus $S_p(n, k)$ is the pth iterated sum, the sum of the sum of . . . the sum of the first n perfect kth powers.

Show that $S_p(n, k)$ is a polynomial in n of degree $p+k$ and determine the form of the polynomial for $k = 3, 4, 5$. If possible, determine the form for general k. (The case $k = 2$ is the subject of problem 22, *Mathematics Magazine*, vol. XXII, no. 1, p. 51.)

4381. *Proposed by R. J. Walker, Cornell University*

The inverse square law has the property that the attraction, at an external point, due to a sphere of uniform density is the same as if the sphere were concentrated at its center. Are there any other laws of attraction which have this property?

4382. *Proposed by E. P. Starke, Rutgers University*

Let $f_1(x)$ be Riemann integrable in the interval $0 \leqq x \leqq M$ and let

$$f_{n+1}(x) = \int_0^x f_n(x)dx, \qquad n = 1, 2, \cdots.$$

Show that

$$\phi(x) = \sum_{n=1}^{\infty} f_n(x)$$

is defined and continuous in the interval except perhaps at discontinuities of $f_1(x)$, and find a simple expression for $\phi(x)$.

4383. *Proposed by Robert Steinberg, University of California, Los Angeles*

Let n symbols be ordered in two different ways, and let a_{ij} denote the number of symbols common to the first i in the first ordering and the first j in the second ordering. Prove that the determinant $|a_{ij}|$ is 1 if the transition from the first ordering to the second is effected by an even permutation and -1 if it is effected by an odd permutation.

4384. *Proposed by H. F. Sandham, Trinity College, Ireland*

In the following modification of the harmonic series there is a change of sign after the reciprocal of each square. Prove that

$$\frac{1}{1}-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}-\frac{1}{10}-\frac{1}{11}-\frac{1}{12}\cdots$$

$$=\sum_{n=1}^{\infty}\frac{\pi}{\sqrt{n}\sinh\pi\sqrt{n}}.$$

SOLUTIONS

Twelve Point Spheres

4265 [1947, 479]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

If two tetrahedra are homothetic with respect to their common centroid, the twelve point sphere of one of these tetrahedra is tangent to the twelve point spheres of the four tetrahedra which the planes of its faces cut off from the trihedral angles of the other tetrahedron.

*Solution by the Proposer.** Let $T\equiv ABCD$ and $T'\equiv A'B'C'D'$ be the two given tetrahedra, homothetic with respect to their common centroid G. Since the trihedral angles (A) and (A') are homothetic with respect to G, the line GAA' contains the centroids G_a and G_a' of the homologous faces BCD and $B'C'D'$. Also the centroid of the triangle $B_1C_1D_1$ cut from the plane BCD by the trihedral angle (A') coincides with G_a. Hence the triangles BCD and $B_1C_1D_1$ and the tetrahedra T and $A'B_1C_1D_1$ are homothetic with respect to G_a (provided that the homothetic ratio for center G is not $-\frac{1}{3}$). The twelve point spheres of T and $A'B_1C_1D_1$ are also homothetic with respect to G_a. The twelve point sphere of T passes through G_a, hence the twelve point sphere of $A'B_1C_1D_1$ is tangent to that of T at G_a.

Note. The second twelve point spheres of two orthocentric tetrahedra homothetic with respect to their common orthocenter have an analogous property.

Nine-point Circle and Artzt Parabolas

4294 [1948, 254]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

The lines joining the orthocenter of a triangle to the points of intersection of the medians with the nine-point circle (other than the midpoints of the sides),

* Translated by W. E. Byrne, Virginia Military Institute.

pass through the vertices of parabolas tangent to two sides of the triangle and having the third side for chord of contact.

Solution by L. M. Kelly, Michigan State College. We use the following notation: A_i are the vertices of the triangle, M_i are the midpoints of the sides, H_i the feet of the altitudes, H the orthocenter, P_i the specified intersections of the medians with the nine-point circle. Let the equation of the parabola be $y=kx^2$. Using familiar properties of the parabola we know that (a) the median A_3M_3 is parallel to the axis of the parabola, (b) the line M_1M_2 is tangent to the parabola, and (c) the orthocenter of the triangle $A_3M_1M_2$, circumscribed to the parabola, is on the directrix. Since the triangle $A_3M_1M_2$ is homothetic to triangle $A_1A_2A_3$ in the ratio 1/2, the orthocenter K_3 of $A_3M_1M_2$ is at the midpoint of the segment A_3H. This point is also on the nine-point circle, as are P_3, M_3, and H_3. Thus $K_3P_3M_3H_3$ is a cyclic quadrilateral and K_3P_3 is therefore perpendicular to the axis, that is, P_3K_3 is the directrix. The coördinates of P_3 are thus easily seen to be $(x_1+x_2)/2$, $-1/4k$.

The coördinates of the orthocenter are now computed in a straightforward fashion and are seen to be $(x_1+x_2)(2k^2x_1x_2+1)$, $-(2k^2x_1x_2+1)/2k$. Thus the slopes of the lines P_3O and HO are both equal to $-1/2k(x_1+x_2)$, so that P_3, O, and H are collinear as required.

Also solved by J. A. Bullard, Joseph Langr, and the Proposer.

Editorial Note. Bullard points out how easily the present proposition follows from his Note, *Further properties of parabolas inscribed in a triangle* (1937, 368). The Proposer gives the following further references to the properties of these parabolas, treated for the first time by Artzt, *Programme scolaire du gymnase de Recklinghausen* (1884): E. Lemoine, P. Barbarin, *Mathesis*, 1884, pp. 12, 142; H. Brocard and G. de Longchamps, *Journal de Mathématiques Spéciales*, 1885, pp. 109, 123; A. Bullard, THIS MONTHLY [1935, 606] and [1937, 368]; V. Thébault, *Archives du Grand-Duché* (Luxembourg) 1948. See also problem 4019 [1943, 517].

Automorphism

4295 [1948, 320]. *Proposed by Irving Kaplansky, University of Chicago*

Show that any group with more than two elements admits an automorphism other than the identity.

Solution by R. C. Buck, Brown University. If the group is not Abelian, any element a not in the center gives a non trivial inner automorphism $x\to a^{-1}xa$. If the group is Abelian, and the square of some element is not the identity, then $x\to x^{-1}$ is such an automorphism. If the square of every element is the identity, and the group has more than two elements, then it is the direct sum of cyclic groups of order 2; the mapping induced by interchanging the generators of two of these furnishes the desired automorphism.

Also solved by Roy Dubisch, J. W. Gaddum, Leo Moser, D. J. Newman, C. R. Phelps, Alex Rosenberg, J. A. Zilber, and the Proposer.

Continued Fractions

4296 [1948, 320]. *Proposed by H. S. Wall, University of Texas*

It is known that when the continued fraction

$$x = p - \cfrac{q}{p - \cfrac{q}{p - \ddots}}$$

converges, then its value is the numerically larger root of the equation $x^2 - px + q = 0$. On the other hand, Newton's formula for computing the roots by successive approximation is

$$x_{k+1} = x_k - \frac{x_k^2 - px_k + q}{2x_k - p} = \frac{x_k^2 - q}{2x_k - p}, \qquad k = 0, 1, 2, \cdots.$$

Show that if x_0 is an approximant of the continued fraction, then $x_1, x_2, x_3, \cdots$ are approximants of the continued fraction.

I. *Solution by J. S. Frame, Michigan State College.* The successive convergents P_n/Q_n to the continued fraction

$$x = a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \ddots}}$$

are most easily displayed by the use of matrices. We may prove by induction that

$$\begin{pmatrix} P_n & b_nP_{n-1} \\ Q_n & b_nQ_{n-1} \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & b_n \\ 1 & 0 \end{pmatrix}, \qquad n > 0,$$

if we define $P_0 = 1$, $Q_0 = 0$, so that $P_1/Q_1 = a_1$.

In the present example where $a_i = p$, $b_i = -q$, we have

$$\begin{pmatrix} P_n & -qP_{n-1} \\ Q_n & -qQ_{n-1} \end{pmatrix} = \begin{pmatrix} p & -q \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} p & -q \\ 1 & 0 \end{pmatrix}\begin{pmatrix} P_{n-1} & -qP_{n-2} \\ Q_{n-1} & -qQ_{n-2} \end{pmatrix}.$$

Hence, $P_n = pP_{n-1} - qQ_{n-1}$, $Q_n = P_{n-1}$ and

$$\begin{pmatrix} P_n & -qQ_n \\ Q_n & P_n - pQ_n \end{pmatrix} = \begin{pmatrix} p & -q \\ 1 & 0 \end{pmatrix}^n.$$

Squaring this we have

$$\begin{pmatrix} P_{2n} & -qQ_{2n} \\ Q_{2n} & P_{2n}-pQ_{2n} \end{pmatrix} = \begin{pmatrix} P_n & -qQ_n \\ Q_n & P_n-pQ_n \end{pmatrix}^2,$$

and expanding, we find

$$\frac{P_{2n}}{Q_{2n}} = \frac{P_n^2 - qQ_n^2}{Q_n(2P_n - pQ_n)}.$$

Hence if we set $x_k = P_n/Q_n$ in Newton's Method, we find $x_{k+1} = P_{2n}/Q_{2n}$. Thus each successive application of Newton's method is equivalent to doubling the length of the continued fraction.

If, however, we obtain x_{k+1} from x_k by the Variation of Newton's Method described in this MONTHLY [51 (1944), 36–38; 52 (1945), 212–214; 55 (1948), 90–94] then we find $x_{k+1} = P_{3n}/Q_{3n}$, so that each application of this method is equivalent to tripling the length of the continued fraction.

II. *Solution by Aaron Herschfeld, Washington, D. C.* Explicit formulas for $u_n = P_n/Q_n$ are given in various books (e.g. C. Smith, *A Treatise on Algebra*, 1892, p. 474; Barnard and Child, *Higher Algebra*, 1936, p. 393) where it is shown that if a and b are the roots of the equation $x^2 - px + q = 0$, then

$$u_n = (a^{n+1} - b^{n+1})/(a^n - b^n)$$

or $= a(n+1)/n$, according as $a \neq b$ or $a = b$. Using the relations $a+b=p$ and $ab=q$ it is easy to verify that

$$(u_n^2 - q)/(2u_n - p) = u_{2n}.$$

Hence if x_0 is an approximant, say u_n, then $x_1 = u_{2n}$, $x_2 = u_{4n}$, $\cdots$.

Also solved by R. C. Buck, V. F. Ivanoff, Roger Lessard, W. T. Scott, and the Proposer.

Symmetric Functions of 1, 2, $\cdots$, n

4297 [1948, 320]. *Proposed by Paul Erdös, Syracuse University*

Put $\phi_k^{(n)} = \sum r_1 r_2 \cdots r_k$, where the r's run through all integers $\leqq n$, and the r's are all different. $\phi_n^{(n)} = n!$ It is well known that $\phi_{n-1}^{(n)} \not\equiv 0 \pmod{\phi_n^{(n)}}$. Prove that if n is sufficiently large $\phi_k^{(n)} \not\equiv 0 \pmod{\phi_n^{(n)}}$ for any k. (This is not always true; e.g., $n=3$, $k=1$).

Editorial Note. As noted by Leo Moser, the present problem is equivalent to Theorem 1 of the paper, *Some Properties of Partial Sums of the Harmonic Series*, by Erdös and Niven (Bulletin of the American Mathematical Society, v. 52, 1946, pp. 248–251.) The editors regret the late date at which the present proposal was printed since its submission by the Proposer actually antedated the Bulletin article by several years.

A detailed solution by Fritz Herzog gives essentially the same solution as that in the cited paper, except for the use of the prime number theorem in a somewhat different form.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y. and not to any of the other editors or officers of the Association.

Analytic Geometry. By Robin Robinson. New York, McGraw-Hill Book Company, Inc., 1949. 143 pages. $2.25.

This book is one of many that has appeared of recent date to serve in the capacity of a concise text covering the field of analytic geometry. In each case the authors have decided for themselves as to just where the brevity is to be effected in their texts. The author of this book sets forth three objectives, namely:

1. To provide a background for the calculus; but, at the same time, stress the geometric properties of the loci discussed.
2. To present each of the topics with a minimum amount of material so as not to steal the show from the teacher and thus permit him to develop the subject in his own way.
3. To provide an ample supply of well chosen problems so that students of all calibers may be able to improve their backgrounds and efficiency with the subject.

The reviewer is of the opinion that for the topics included in this *Analytic Geometry* the author has partially accomplished his aims, but has failed to do so completely because of the omission of certain vital topics essential for making analytic geometry the necessary background for the calculus which follows. Such topics as transcendental, exponential, higher plane algebraic and parametric equations are omitted from consideration in this text, and as a consequence it would fail in its most important aim of being a background for the calculus.

There are other minor criticisms such as, for example, the confusing use of the term common chord interchangeably with that of radical axis; the omission of the normal form for the equation of a straight line; and, the omission of the study for the angular bisector locus between two lines.

This text at best gives an above average treatment of analytic geometry up to and including the work on the conic section in the plane; and a standard introduction to space analytics up to and including quadric surfaces. The publishers should be commended for the excellent format of the book which has made whatever contents it does contain attractive in appearance.

S. SELBY

First Year College Mathematics with Applications. By P. H. Daus and W. M. Whyburn. New York. The Macmillan Company. 1949. 13+495 pages. $5.00.

The authors say in the preface, "This book is written with the intention of providing a single text for the study of first year college mathematics, especially

in engineering and technical schools. Its purpose is to provide a strong and ample background for the study of the calculus, and to integrate the subjects of college algebra, analytic geometry, and analytic trigonometry. At the same time the text illustrates all principles by applications taken from science and engineering, so that the course is completely independent of its future use. . . . Suggested lesson outlines for a ninety-hour course (three days a week for two semesters) and for a seventy-five-hour course are given. . . . "

Those who prefer to give relatively strong training in analytic geometry should welcome this book with enthusiasm. In sharp contrast with recent works which combine analytics with calculus and thereby effect a de-emphasis of the former subject, this text retains most of the traditional geometrical topics. Moreover, it provides a vigorous and well-motivated presentation which, in the opinion of the reviewer, is superior to that of many established books. The authors have compiled an extensive list of exercises many of which involve applications to engineering and scientific problems. Numerous sections are devoted to specific applications, for example—Electrical Resistance; Beam Equilibrium; Shear and Moment Diagrams; Electrical Networks; Suspension Bridges, Road Cross-sections; Bending Moments; Vibrating Spring; Vector Impedance; Voltage and Current; Series Circuits; Divided Circuits; Damped Harmonic Functions; Lissajous Figures. Of course these may be omitted with out destroying continuity. By including a fair number of these sections, the instructor can introduce his students to several topics normally met in more advanced engineering work. In this way, the authors have furnished ample motivation for study of nearly all material included.

The student is assumed to be proficient in the purely manipulative aspects of algebra. Although the quadratic equation is discussed at some length, little is done with other topics (progressions, binomial theorem, logarithmic computation, *etc.*) normally handled in lower courses. Topics of questionable direct value to engineering students, such as permutations and combinations, classical probability, and mathematics of finance are omitted. Otherwise, most of the traditional college algebra content is included, although some topics such as inequalities are treated only as tools in geometric investigations. In connection with irrational roots of equations, linear and parabolic interpolation and Horner's Method are discussed. Solution of systems of linear equations is accomplished by elimination with an account of the Doolittle Method included. This is in line with the opinion of many engineers and mathematicians that, in practice, methods involving determinants are often not the most desirable. There are sections on hyperbolic, logarithmic and exponential functions and on solutions of transcendental equations.

The reviewer is pleased with the care exercised by the authors in connection with curve tracing where the single-valuedness of the radical is emphasized. He is less pleased with the many-valued interpretation of the same symbol in connection with roots of complex numbers. However, the whole treatment of these topics is certainly not inferior to that of most texts at this level.

In analytics, the straight line and the parabola play leading roles. The various forms of the linear equation are derived in a straight-forward way based upon the concept of slope. In the normal form discussion, the angle between the horizontal axis and the normal is always taken in the first or second quadrant. The positive side of the line is then the upper half-plane.

The treatment of the problem of curve fitting is extensive and well-done. There are sections on the method of moments, the logarithmic papers, least squares, and parabola of regression. Parallel scales and N-charts are discussed. The reviewer did not notice the omission of any topic in plane analytics which is generally considered as appropriate for study by the prospective engineer.

Two chapters are devoted to analytic trigonometry. These are rich in applications and the last chapter contains a superior discussion of polar coordinates. Computational trigonometry is assumed to have been previously covered by the student.

Each set of exercises begins with several labelled "oral." These should be of considerable help to the instructor in directing classroom discussion along profitable lines. The sections on special applications mentioned above will aid the inexperienced instructor as well as the student. The book seems to contain ample material and to be teachable subject to the assumption that the student is at least moderately well prepared. Persons in charge of freshman engineering courses certainly should give consideration to this text.

E. M. BEESLEY

Introduction to Complex Variable and Applications. By R. V. Churchill. New York, McGraw-Hill, 1948. 6+216 pages. $3.50.

This is a text in functions of a complex variable that should prove very popular with students of mathematics, physics and engineering at the senior or first year graduate level. While it is not a treatise on the subject, this book does present logically the parts of complex variable theory necessary for the applications made in the text. Other applications, especially of the theory of residues and in the evaluation of integrals, are to be found in the same author's *Modern Operational Mathematics in Engineering*. These two, with an earlier text, *Fourier Series and Boundary Value Problems*, form a set, closely related in purpose, presenting three important classical methods for the solution of boundary value problems: the separation of variables, the operational method, and now conformal mapping.

In the first four chapters of this book, the author introduces complex numbers and analytic functions, and makes a detailed study of the elementary functions. Then come chapters on integration, power series and residues. Following this are three chapters on conformal mapping and its applications, and the last two chapters are devoted to analytic continuation and Riemann surfaces.

Emphasis is placed on teaching the subject rather than on its development, although there is no sacrifice of logic in the development. The author does not use a standardized procedure in introducing and proving theorems, but states

theorems sometimes before, sometimes after the proof has been made, intending thereby to improve the clarity or preserve the continuity of his development. This technique is one that is carried over from his earlier texts, and helps to keep the wording clear, concise and yet unstrained.

Problems are usually chosen to illustrate points made in the text, with very few designed to extend the theory. In some cases this practice leads to many tedious drill problems, as in the exercises relating to the circular and hyperbolic functions, where proofs of numerous formulas are called for. Such specific problems, however, have more meaning for the student than those calling for an extension of work in the text.

Direct applications of the theory are made in the chapters on conformal mapping and the Schwarz-Christoffel transformation, where problems in steady-state temperature, electric potential and incompressible fluid flow are solved. The solution of many of these problems is facilitated by an illustrated table of conformal transformations in the appendix. Some applications of the theory of residues to the evaluation of integrals are also given. More of these are to be found in *Operational Mathematics*.

This book should be valuable as a text for a one semester course in complex variable, where emphasis on applications of conformal mapping and the theory of residues is desired. Answers to many, but not all, of the problems are given.

Very few minor errors were encountered, all of a typographical nature.

R. E. GASKELL

NEW BOOKS RECEIVED

Non-linear Problems in Mechanics of Continua. (Proceedings of Symposia in Applied Mathematics, vol. 1.) New York, American Mathematical Society, 1949. 8+219 pp. $5.25.

Regular Polytopes. By H. S. M. Coxeter. New York, Pitman Publishing Corp., 1947. 19+321 pp.

Mathematics for Finance and Accounting. By Coleman and Rogers. Pitman Publishing Corporation. 1949. $4.00

The Aims of Education and Other Essays. By A. N. Whitehead. New York, New American Library, 1949. 168 pp. $.35.

Analytic Geometry. Third Edition. By W. A. Wilson and J. I. Tracey. New York, Heath, 1949. 10+318 pp. $2.75.

Geometrical Tools. Revised Edition. By R. C. Yates. St. Louis, Educational Publishers, 1949. 194 pp.

University Mathematics. By J. Blakey. London, Blackie and Son, 1949. 8+527 pp. 25 s.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

CLUB REPORTS, 1948–49

Pi Mu Epsilon, Lehigh University

During the past academic year the *Pennsylvania Gamma* chapter of *Pi Mu Epsilon* at Lehigh University has enjoyed five meetings and an initiation banquet.

Papers presented during the year were as follows:

The Laplace transform, by Prof. R. Stoll

The Laplace transform, a second paper, by J. Harry Dowling

Some approximation methods, by Prof. C. Shook

The deviation of the Lorentz equation, by Mr. Alperin

Some boundary value problems. by Prof. Beer.

At the initiation banquet Prof. A. Pitcher presented a very interesting talk on topology entitled, *The ham sandwich problem.*

New officers were elected for the coming year as follows: Director, Prof. R. Stoll; President, Richard Orford; Vice-President, James Fulleylove; Secretary, Albert Christy; Treasurer, Charles Close.

Kappa Mu Epsilon, Mount Mary College

The *Wisconsin Alpha* chapter of *Kappa Mu Epsilon* held monthly meetings, at which the following papers were given by the members:

Mathematics and Aristotle, by Norma Harding

What about Copernicus?, by Betty Prossen and Rosemary White

Descartes, by Dorothy Vande Walle

Galileo, by Kathleen Hanley

Napier, by Mary Suehr

Trigonometry, by Mary Hunt

The Pythagorean theorem, by Wanda Kropp.

Officers for 1948–49 were: President, Norma Harding; Vice-President, Bernadine Spitznogle; Secretary, Mary Alice Gauerke; Treasurer, Marilyn Briggemann; Corresponding Secretary, Sister Mary Felice.

Dorothy Karner was elected President for the year 1949–50.

Mathematics Colloquia, Purdue University

The following lectures were presented to the members during the year:

The modern concept of infinity and *The topology of general function spaces*, by Prof. C. Kuratowski

Non-cut points, by Prof. K. Zarankiewicz

Grassmann algebras and determinants and *Cohomology theory of abelian groups*, by Prof. S. MacLane

On the mean value of an analytic function, by Dr. H. Shniad

Continuous spectra of quadratic forms and integral equations, by Prof. E. Hellinger

The application of differential equations to the derivation of sampling distributions, by Prof. O. Reiersol

Coefficients of Schlicht functions, by Prof. A. Schaiffer

Determinantal congruences in algebraic number fields, by Prof. R. Hull.

Pi Mu Epsilon, Montana State University

The following papers have been presented to the *Montana Alpha* chapter of *Pi Mu Epsilon* since its reorganization in February 1949.

Careers in science, by Prof. G. Shallenberger

Mathematics and chemistry, by Ted Burton

New absorption bands of the hydrogen molecule, by Paul Pfleuger.

Pi Mu Epsilon entrance prizes of twenty-five, fifteen, and ten dollars, given on the basis of an examination offered to freshmen from Montana High Schools, were awarded this year to Mr. Thronson, first prize; Mr. Wardahl, John Pecarich, and Keith Wilson, all tied for second prize.

Officers for the coming year are: Director, John Peterson; Vice-Director, Donald Philips; Secretary, Donald Marshall; Treasurer, Prof. T. Ostrom.

Kappa Mu Epsilon, Upsala College

During the first semester, the members studied *nomography*. By direction of the sponsor they followed a basic text and collateral reading, and at three successive meetings the material was discussed by Frances Rischmuller, Martin Monroe and James Gill.

During the second semester, the theme was *famous problems in geometry and analysis*. Each topic was prepared in the form of a seminar paper, and included the mathematical background and history of the problem, its ancient and more modern solutions, and its practical applications, both ancient and modern. The papers presented were:

Symmetric functions, by Edward Pulchlopek

The quadrature of the circle, by Robert Schenk

The duplication of the cube, by John Armstrong

The trisection of the general angle, by Robert Warnken

Axioms and postulates, by Robert Wallace

Hyperbolic functions, by Martin Netzler

The cycloid, by Jerome Drexler.

Other addresses heard were:

Zeno of Elia, by Dr. C. Vedova, of Newark College of Engineering

What's in a name? a talk on mathematical nomenclature, by Dr. C. Boyer, of Brooklyn College.

The officers for the year 1949–50 are: President, Victor Valentino; Vice-President, Hannah Magill; Secretary, Ethel Larson; Treasurer, Robert Ruppert; Historian, Hannah Magill; Corresponding Secretary, Prof. Martin Nordgaard.

Mathematics Club, Haverford College

The following papers were presented before the regular meetings of the Haverford *Mathematics Club*:

Functional equations, Prof. D. L. Thomsen, Jr.

The mathematics of music, by J. E. Brownlee

Fibonacci and Farey series with applications, by R. T. Ives.

Problems for which the Mathematics department offered prizes, were discussed at each meeting.

Prof. Kasimir Kuratowski, Visiting Lecturer of the American Mathematical Society from the University of Warsaw, Poland, was a guest of the college for a day. He talked informally with the students and delivered an address on *Modern concepts of infinity*.

Prof. C. O. Oakley served as Faculty Advisor of the club.

Kappa Mu Epsilon, Mount St. Scholastica College

Papers presented to the *Kansas Gamma* Chapter of *Kappa Mu Epsilon* for 1948–49 were:

Mathematics related to the modern world, by Gloria Jaskowiak

Empirical and theoretical statistics, by Mary Alice Weir

A survey of the history and development of non-euclidean geometries, by Sister Evangeline Anderson, O.S.B.

Modern men of mathematics, by Jeri Sullivan, Mary J. Martin, Frances Donlon, and Sally Crisanti

Modern women mathematicians, by Noreen Hurter, Gertrude Harrison, and Jeanne Culivan

Nomography for science students, by Mary Alice Weir.

The recipient of the Hypatiean Award, for her outstanding and original research on *Nomography*, was Mary Alice Weir. The award is made each year to the member making an outstanding contribution to the Chapter.

The Chapter, together with the mathematics department, presented an assembly for the school body centering around the application of mathematics in the life of the student. It included *A Mathematical Playlet*, produced by the pledges showing the application of mathematics in the various fields of knowledge, a choral reading *An Ode to the Queen of Sciences* and a discussion on the popular history of mathematics.

Newly elected officers are: President, Noreen Hurter; Vice-President, Mary Alice Weir; Secretary, Jeanne Culivan; Treasurer, Jeri Sullivan; Musician, Evelyn Wagner; Publicity Chairman, Patti Shideler.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

APPOINTMENTS AVAILABLE AT THE INSTITUTE FOR NUMERICAL ANALYSIS

Appointments of the following two types are being offered to properly qualified graduate students in pure and applied mathematics at the Institute for Numerical Analysis of the National Bureau of Standards at the University of California, Los Angeles.

(1) *Summer Studentships*

The purpose of these appointments is to introduce promising graduate students in mathematics and related fields to the work of the Institute for Numerical Analysis through the medium of an intensive program of mathematical seminars and courses in the use of modern computing equipment, and through active participation in the work of the Institute. The program lasts for ten weeks during the summer, from the middle of June to the end of August. Stipends range from $500 to $700.

(2) *Thesis Fellowships*

The purpose of these appointments is to provide the Institute for Numerical Analysis with a limited number of talented doctoral candidates whose research program, in the opinion of the Director of Research of the Institute, is relevant to the work of the Institute. A candidate for such a fellowship must have completed all residence and language requirements for the Ph.D. degree at an accredited university and must obtain the permission of the university to write the thesis in residence at the Institute. Preference is given to a candidate who has selected his thesis topic and has made some progress on the thesis prior to accepting the fellowship. Holders of these fellowships may absent themselves from the Institute for the purpose of taking necessary examinations, but travel money for such purposes will not be authorized. Additional course work may be taken at the University of California, Los Angeles, or elsewhere, but must be kept within a limit of six semester hours in each semester. The holder of a thesis fellowship will ordinarily not be permitted to accept additional employment for remuneration. If the thesis is published, it is expected that a credit line to the Institute and its financial sponsor, the Office of Naval Research, will be incorporated into the article, presumably as a footnote. The stipend for a thesis fellowship is at present fixed at approximately $2,000 for an 11-month working year, provided that the candidate already possesses an M.A. degree or equivalent experience. A thesis fellowship appointment is normally for one year only, but if circumstances warrant, the appointment may be extended for a period not to

exceed one additional year. Such an extension must receive the prior approval of the Civil Service Commission.

Five thesis fellowships are at present assigned by the Bureau to the Institute for Numerical Analysis.

All applicants should clearly understand that the central function of the Institute is to perform mathematical research and exposition pertinent to the effective use of automatic digital computing machinery. As supplementary to this main function, the Institute supports certain research projects in pure and applied mathematics which, while they do not directly bear upon the development of numerical analysis, nevertheless require extensive computations of a type for which automatic machinery is suitable. In other words, the work of the Institute concerns the development and application of the art and science of computing, and no attempt is made to conduct a well-rounded, general program of mathematical research and instruction.

In the case of Summer Studentships and the Thesis Fellowships, a preliminary application must be made to the Director of Research of the Institute for Numerical Analysis before the first of March preceding the summer or academic year for which the fellowship is requested. This preliminary application is to consist of: (a) a letter from the applicant to the Director of Research indicating his interests and experience; (b) a transcript of the applicant's undergraduate and graduate collegiate record to date; and (c) two supporting letters addressed to the Director of Research from scientists of established reputation who are familiar with the work of the applicant.

Appointees will in general be selected on the basis of the preliminary applications. The chosen candidates will be sent Civil Service Forms 57 to complete, and the formalities of appointment will thereafter be those of regular Civil Service recruitment. The awards of these fellowships will be made on or before April 1.

Other appointments at the Institute are made through regular Civil Service channels. There will be available, from time to time, positions on the staff suitable for younger mathematicians and physicists who have already obtained their doctorates, or who are of comparable standing.

CARE'S BOOK PROGRAM

The Book Program of CARE has been developed to aid in the educational and cultural reconstruction of foreign countries. A professional committee, headed by Dr. Luther Evans, Librarian of Congress, has selected 1300 titles in various categories. CARE is equipped to handle the purchasing and shipping of these books to institutions that need them. At present the list of countries to be served includes Austria, Belgium, Czechoslovakia, Finland, France, the American and French Zones of Germany and Berlin, Greece, Italy, Japan, the Netherlands, Norway, and the United Kingdom.

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PERSONAL ITEMS

Professor G. T. Whyburn of the University of Virginia has received an honorary degree of Doctor of Science from Washington and Lee University.

Brooklyn College reports that Assistant Professor Harriet Griffin has been promoted to an associate professorship and that Dr. M. E. Levenson, formerly of Cooper Union, has been appointed to an instructorship.

Columbia University announces: Professor Claude Chevally of Princeton University has been appointed to a professorship; Professor Edward Kasner has retired with the title of Adrain Professor Emeritus.

At Montana State College: Mr. Adrien Hess has been promoted to an assistant professorship; Mr. Robert Glauz, Mr. William Swartz, Mr. J. E. Whitesitt have been appointed to instructorships.

New York University reports: Instructors J. van Heijenoort and John Schoonmaker have been promoted to assistant professorships; Dr. G. E. Noether has been appointed to an instructorship; Professor H. R. Cooley has been on sabbatical leave for the first semester 1949–50.

Ohio State University announces the following promotions and appointments: Associate Professor R. G. Helsel has been named Acting Chairman of the Department of Mathematics; Dr. O. W. Rechard and Dr. R. L. Swain have been promoted to assistant professorships; Dr. E. D. Cashwell of the University of Wisconsin has been appointed to an instructorship.

Ohio University reports the appointments of Mr. John Ridout of the University of Missouri to an acting assistant professorship and Mr. John Hokanson, formerly graduate student at the University of Michigan, to an acting instructorship.

Oregon State College announces the following promotions: Associate Professors C. L. Clark and A. T. Lonseth to professorships and Assistant Professors J. C. R. Li and A. R. Poole to associate professorships.

The Stout Institute reports the promotion of Assistant Professor F. E. Tustison to a professorship and the appointment of Mr. Mathew Renneson to an instructorship.

State College of Washington announces: Professor E. C. Colpitts has retired; Dr. J. L. Brenner has been appointed to an assistant professorship; Dr. P. A. Clement has been appointed to an instructorship.

At the University of Chicago a Conference on Topology will be held during the Spring of 1950; more details will be announced later. The following appointments and promotions have been made: Assistant Professor P. R. Halmos has been promoted to an associate professorship; Mr. A. F. Strehler, formerly assistant at the University of Wisconsin, has been appointed to an instructorship. Professor M. H. Stone, chairman of the Department of Mathematics lectured in Tokyo, Osaka, Kyoto and Nagoya, Japan, during September, 1949 as well as

at the University of Indonesia and in Singapore; Professor Stone is now in India and is giving a course of lectures at the Tata Institute of Fundamental Research in Bombay where he will spend several months. Professor A. A. Albert will be Acting Chairman during the Autumn and Spring Quarters of 1949–50 and Professor Saunders MacLane during the Winter Quarter, 1950.

The University of Maine reports the appointments of Instructors Nelma K. Ebbeson and R. N. Mathias.

The University of Michigan makes the following announcements: Assistant Professors E. H. Rothe and Hans Samelson have been promoted to associate professorships; Instructor Leonard Tornheim has been promoted to an assistant professorship; Professor Fritz Herzog of Michigan State College has been appointed Visiting Associate Professor; Dr. D. A. Darling of Rutgers University, Dr. W. J. LeVeque of Harvard University, Mr. A. J. Lohwater, Dr. Imanuel Marx, Dr. C. J. Titus of Syracuse, and Dr. J. L. Ullman have been appointed to instructorships; Professor R. L. Wilder is on sabbatical leave for 1949–50 and is located at California Institute of Technology; Professor R. V. Churchill was on leave for the first semester, 1949–50; Professor S. B. Myers is on leave for the second semester 1949–50; Dr. E. E. Moise and Dr. M. A. Woodbury are on leave for the academic year.

At the University of Nebraska: Professor M. G. Gaba has retired with the title of Professor Emeritus; Mr. C. C. Buck, University of Michigan, has been appointed to an instructorship.

The University of North Carolina announces the promotion of Lecturer I. R. Hershner, Jr., to an assistant professorship and the appointment of Mr. W. R. Mann to an instructorship.

The University of Notre Dame reports: Assistant Professors Ky Fan and J. P. LaSalle have been promoted to associate professorships; Reverend F. L. Brown and Reverend H. F. DeBaggis have been promoted to assistant professorships; Mr. J. W. Riner and Mr. James C. Smith, Jr., have been appointed to instructorships; Reverend H. F. DeBaggis is on leave of absence for the academic year 1949–50 and is located at Princeton University.

At the University of Tennessee the following have been appointed to instructorships: Mr. E. L. Eagle, University of Arkansas, Mr. A. A. Ritchie of Central State College and Dr. Annette Sinclair of the University of Illinois.

Professor C. B. Allendoerfer of Haverford College has been appointed Visiting Professor of Mathematics at the Massachusetts Institute of Technology for the second semester, 1949–50.

Mr. D. L. Barrick of the University of Colorado has been promoted to an assistant professorship.

Assistant Professor R. V. Blair of Vanderbilt University has received a promotion to an associate professorship.

Dr. C. E. Bures has been appointed Assistant Professor of Philosophy and Psychology at the California Institute of Technology.

Professor H. C. Christofferson, who has been serving in the educational di-

vision of Military Government, Wiesbaden, Germany, has returned to his position as acting director of teacher training and professor of mathematics at Miami University.

Mrs. Joan R. Clark has accepted a position as research assistant with the Institute for Cancer Research, Philadelphia.

Associate Professor W. A. Gager of the University of Florida has been promoted to a professorship.

Associate Professor H. A. Giddings, University of New Hampshire, has been appointed to a professorship in the College of Engineering, New York University.

Assistant Professor E. L. Godfrey of Defiance College has been promoted to an associate professorship.

Assistant Professor F. M. Goen of Mississippi State College has been promoted to an associate professorship.

Dr. D. B. Goodner of the University of Illinois has been appointed to an assistant professorship at Florida State University.

Dr. Verena Haefeli has been appointed to an instructorship at Goucher College.

Assistant Professor A. E. Hallerberg of Illinois College has been promoted to an associate professorship.

Professor W. K. Hayman, University College of the Southwest of England, Exeter, has been appointed Lecturer in Mathematics for 1949–50 at Brown University.

Associate Professor W. P. Heinzman of New Mexico College of Agricultural and Mechanic Arts has been promoted to a professorship.

Mr. A. J. Hoffman has been appointed to an instructorship at Barnard College, Columbia University.

Professor Eberhard Hopf has been appointed to a professorship at New York University.

Instructor R. J. Howerton is now Acting Head of the Department of Mathematics of Regis College.

Mr. T. C. Hutchison has accepted a position as electrical engineer with Haller, Raymond & Brown, Inc., State College, Pennsylvania.

Associate Professor H. H. Irwin of State College of Washington has been promoted to a professorship.

Professor R. D. James of the University of British Columbia has been promoted to the position of Head of the Department of Mathematics.

Assistant Professor Free Jamison has been promoted to an associate professorship at San Jose State College.

Mr. H. J. Jones has been appointed to an instructorship at El Camino College, California.

Dr. J. J. Kinsella, formerly at Ohio State University, has been appointed Chairman of the Department of Mathematics Education, School of Education, New York University.

Mr. L. M. Klauber is now Chairman of the Board, San Diego Gas and Electric Company.

Mr. S. L. Levy has been appointed Research Assistant at Brown Universty.

Professor Z. L. Loflin, previously Acting Head of the Department of Mathematics of Southwestern Louisiana Institute, has been appointed Head of the Department.

Dr. M. M. Lotkin is now Chief, Machines Section of Ballistics Research Laboratory, Aberdeen Proving Ground.

Dr. C. W. Mathews of Washington University has been promoted to an assistant professorship.

Miss Lida B. May has been promoted to an assistant professorship at Texas Technological College.

Associate Professor M. G. McCuan has been promoted to the position of Chairman, Department of Mathematics and Engineering, Amarillo College.

Reverend J. L. McKenney has been promoted to the position of Head of the Department of Mathematics of Providence College.

Mr. M. M. McLynn, previously a student at George Washington University, has accepted a position as mathematician with Engineering Research Associates, Arlington, Virginia.

Mr. S. S. McNeary has been promoted to an associate professorship at Drexel Institute of Technology.

Professor W. M. Miller, formerly head of the Department of Mathematics of Robert College, Istanbul, Turkey, has been appointed to a professorship at Southwestern Louisiana Institute.

Mr. W. H. Mills has been appointed to an instructorship at Yale University.

Associate Professor W. L. Moore of the University of Louisville has been promoted to a professorship.

Assistant Professor R. H. Moorman, Tennessee Polytechnic Institute, has been promoted to an associate professorship.

Associate Professor H. T. Muhly of the United States Naval Academy has been appointed to an associate professorship at the State University of Iowa.

Reverend J. P. Murray has been promoted to an assistant professorship at Fairfield University.

Mr. D. J. Myatt of Clarkson College of Technology has been appointed to an instructorship at Antioch College.

Mr. F. P. Nash is now Head of the Department of Mathematics of Groton School.

Assistant Professor C. G. Peckham of the University of Dayton has been promoted to an associate professorship.

Mr. R. P. Peterson, Jr., of the University of California at Los Angeles has a position as mathematician at the Institute for Numerical Analysis of the National Bureau of Standards, Los Angeles.

Mr. O. L. Phillips, who has been serving as Acting Head of the Department

of Mathematics of Mississippi Southern College, has been promoted to the position of Head of the Department.

Dr. W. L. Pickard, formerly chairman of the Department of Mathematics, University of Illinois, Galesburg Division, has been appointed to the position of Professor of Mathematics at the State University of New York at New Paltz.

Mr. Frank Pomila has been appointed to an instructorship at St. John's University.

Associate Professor R. M. Rankin has been promoted to the position of Professor of Mathematics and Chairman of the Department of Mathematics at Missouri School of Mines.

Mr. Russell Remage has returned to the University of Delaware as Instructor of Mathematics after a year at the University of Pennsylvania.

Assistant Professor H. L. Rice, University of Omaha, has been promoted to an associate professorship.

Assistant Professor R. A. Rosenbaum of Reed College has been promoted to a professorship.

Professor W. E. Roth, formerly at Sampson College, has been appointed to a professorship at the University of Tulsa.

Mr. R. S. Seamons has been appointed to an instructorship at Yakima Valley Junior College, Washington.

Assistant Professor C. L. Seebeck, Jr., of the University of Alabama has been promoted to an associate professorship.

Mr. R. D. Sheffield has been appointed to an instructorship at the University of Mississippi.

Assistant Professor Jack Silber, Roosevelt College, has been promoted to an associate professorship.

Mr. David Singer has accepted an appointment as structural engineer with M. H. Treadwell Company, New York City.

Sister Jeanette Obrist has been promoted to an associate professorship at Mount St. Scholastica College.

Sister M. Thomas à Kempis, previously head of the Department of Mathematics of the College of St. Teresa, has been appointed Principal of St. Mary High School, Sleepy Eye, Minnesota.

Sister Marie Gertrude is now Chairman of the Department of Mathematics, Seton Hill College.

Mr. Rubin Smulin is an instructor at Frame Refrigeration-Electrical Institute, Miami, Florida.

Mr. B. R. Snyder of the University of New Hampshire has been appointed to an instructorship at Johns Hopkins University.

Professor T. H. Southard of Wayne University has been promoted to an associate professorship.

Dr. Vivian E. Spencer is now Chief, Minerals Section, Bureau of the Census, Washington, D. C.

Instructor F. M. Stein of Iowa Wesleyan College has been promoted to an associate professorship.

Mr. C. S. Stuckey has accepted a position with Standard Register Company, New York.

Mrs. Falka G. Sturges is teaching at Hassler High School, Redwood City, California.

Instructor E. A. Sturley of Allegheny College has been promoted to an assistant professorship.

Miss M. Frances Suter of Roanoke College has been promoted to an assistant professorship.

Mr. A. F. Svoboda has been promoted to an assistant professorship at DePaul University.

Dr. Olga Taussky has accepted a position as mathematician with the National Bureau of Standards.

Instructor Takashi Terami has been promoted to an assistant professorship at the College of St. Thomas.

Mrs. Marian W. Thornton has been appointed to an assistant professorship at the University of Minnesota.

Mr. J. C. Tinner of Florida Agricultural and Mechanical College has been promoted to an associate professorship.

Associate Professor W. R. Transue of Kenyon College has been promoted to a professorship.

Mr. C. W. Trigg of Los Angeles City College has accepted the editorship of the Problems and Questions Department of the Mathematics Magazine.

Dr. G. W. Tyler is now Consultant in Mathematics and Statistics, Naval Electronics Laboratory, San Diego, California.

Mr. J. T. Vallandingham has been promoted to the position of Head of the Department of Mathematics of Cumberland College.

Mr. G. H. Van Arkel has been promoted to the position of Head of the Department of Science and Mathematics, Everett Junior College, Washington.

Dr. D. D. Wall has been appointed to an instructorship at Santa Barbara College, University of California.

Instructor William Wallace of Northeastern University has been promoted to an assistant professorship.

Associate Professor Emily H. Watkins of the Woman's College of the University of North Carolina has been promoted to a professorship.

Dr. Margaret F. Willerding has been promoted to an assistant professorship at Harris Teachers College.

Dr. G. H. Wilson of Drexel Institute has been promoted to an assistant professorship in the Department of Physics.

Assistant Professor Emeritus H. L. Beard of Oregon State College died on May 19, 1949.

Professor Emeritus Talmon Bell of Sterling College died on October 20, 1949. He was a charter member of the Association.

Mr. W. J. Bower of Upsala College died November 14, 1949.

Mr. Herman Karnow, teacher at Brooklyn Technical High School, died December 31, 1948.

Professor Emeritus A. C. Lunn of the University of Chicago died on November 19, 1949. He was a charter member of the Association.

Dr. Lao G. Simons, professor emeritus of Hunter College of the City of New York and a charter member of the Association, died November 25, 1949.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 79 persons have been elected to membership by the Board of Governors on applications duly certified:

C. E. Amos, Ph.D. (Ohio State) Asso. Professor, University of Toledo, Ohio.

M. Q. Barton, Student, Central College, Fayette, Mo.

V. W. Beck, M.S. (Minnesota) Elect. Engr., American Television & Radio, St. Paul, Minn.

H. A. Bernhard, B.S.E.E. (Newark) Grad. Student, Columbia University, New York, N. Y.

W. S. Bicknell, M.S. (Michigan) Instructor, Iowa State College, Ames, Iowa.

R. N. Bradt, M.S. (Colorado State) Asst. Instructor, University of Kansas, Lawrence, Kans.

G. M. Brown, D.Sc. (Michigan) Asso. Professor University of Georgia, Atlanta, Ga.

R. C. Campbell, B.A. (Dartmouth) Grad. Asst., University of Colorado, Boulder, Colo.

Ion Carstoiu, Ph.D. (Paris) Asst. Professor, Johns Hopkins University, Baltimore, Md.

C. C. Chouteau, A.B. (Kansas) Teacher, Bowring High School, Okla.

D. E. Coffey, Ed.B. (Illinois State Normal) Grad. Student, Montana State University, Missoula, Mont.

E. J. Cogan, M.A. (Wisconsin) Instructor, Pennsylvania State College Center, Pottsville, Pa.

H. K. Crowder, M.S. (Case) Instructor, Case Institute of Technology, Cleveland 2, Ohio.

D. L. Daly, M.A. (Michigan) Instructor, University of Kentucky, Lexington, Ky.

H. E. Dickey, M.S. (Iowa State) Asst. Professor, Iowa State College, Ames, Iowa.

M. H. Dipert, M.S. (Indiana) Professor, Westmar College, LeMars, Iowa.

William Douglas, Courtenay, B.C., Canada.

W. E. Dwyer, Jr., Student, Catholic University of America, Washington, D. C.

D. F. Eliezer, B.A. (Lehigh) Grad. Asst., Lehigh University, Bethlehem, Pa.

E. R. Epperson, B.S.Ed. (Miami Univ.) Grad. Asst., Miami University, Oxford, Ohio.

A. G. Fadell, B.A. (Buffalo) Teaching Fellow, University of Buffalo, N. Y.

W. J. Feeney, S.J., M.A. (Boston College) Grad. Student, Catholic University of America, Washington, D. C.

J. F. Foley, B.Sc. (Chicago) Asst. Physicist, Gas and Electric Co., Baltimore, Md.

R. A. Forseth, B.A. (S.T.C., Moorhead, Minn.) Instructor, State Teachers College, Mayville, N. D.

E. J. Germino, M.A. (Columbia) Instructor, St. John's University, Brooklyn, N. Y.

D. B. GOODNER, Ph.D. (Illinois) Asst. Professor, Florida State University, Tallahassee, Fla.

DONALD GREENSPAN, M.S. (Wisconsin) Instructor, University of Maryland, College Park, Md.

H. C. GRIFFITH, A.B. (Missouri) Asst. Instructor, University of Missouri, Columbia, Mo.

B. H. GUNDLACH, Ph.D. (Hamburg) Asst. Professor, University of Arkansas, Fayetteville, Ark.

EDWIN HEWITT, Ph.D. (Harvard) Asst. Professor, University of Washington, Seattle, Wash.

I. I. HIRSCHMAN, JR., Ph.D. (Harvard) Asst. Professor, Washington University, St. Louis, Mo.

A. J. HOFFMAN, A.B. (Columbia College) Instructor, Barnard College, Columbia University, New York, N. Y.

C. W. HUFF, M.S. (South Carolina) Instructor, University of South Carolina, Columbia, S.C.

A. R. HYDE, M.A. (Trinity) Instructor, Trinity College, Hartford, Conn.

W. G. JOHNSON, M.A. (Michigan) Asst. Professor, Phillips University, Enid, Okla.

F. B. JONES, Ph.D. (Texas) Asso. Professor, University of Texas, Austin, Tex.

R. S. KINGSBURY, M.S. (Wisconsin) Staff Member, Operations Evaluation Group, Navy Department, Washington, D. C.

J. J. KINSELLA, Ed.D. (Columbia) Asso. Professor, New York University, Washington Square, New York, N. Y.

A. T. KOVITZ, B.S.E.E. (Purdue) Instructor, Polytechnic Institute of Brooklyn, N. Y.

OLIVE LESKOW, M.A. (Minnesota) Teacher, Tolleston School, Gary, Ind.

T. M. LITTLE, Ph.D. (Maryland) Asst. Professor of Biology, University of Nevada, Reno, Nev.

J. D. MATHESON, B.S. (U. S. Military Acad.) Lt. Col., U. S. Army, Alexandria, Va.

R. L. MAY, B.S. (Webb Inst.) Technical Asst., Socony-Vacuum Oil Co., New York, N. Y.

M. J. F. MCKEON, M.A. (New York S.C.T., Albany) Instructor, Rensselaer Polytechnic Institute, Troy, N. Y.

J. R. MCPHERSON, Student, University of North Dakota, Grand Forks, N. D.

KOVINA MILOSEVICH, Student, University of Skoplje, Yugoslavia.

MARIE A. MOORE, D.Ed. (Colorado) Professor, Harris Teachers College, St. Louis, Mo.

GERTRUDE C. NAUMAN, Student, Bryn Mawr College, Pa.

J. D. NEFF, B.A. (Coe) Grad. Asst., Kansas State College, Manhattan, Kans.

AGNES E. NIBARGER, M.S. (Wichita) Instructor, University of Wichita, Kans.

O. M. NIKODYM, Ph.D. (Warsaw) Professor, Kenyon College, Gambier, Ohio.

V. D. NYHOFF, B.S. (Kansas State) Grad. Asst., Kansas State College, Manhattan, Kans.

D. G. O'CONNOR, B.E.E. (Manhattan) Jr. Engr., Sylvania Electric, Bayside, L. I., N. Y.

LEOLA P. ODLAND, Student, University of Oregon, Eugene, Ore.

ALICE OSTERBERG, A.M. (Columbia) Instructor, Brooklyn College, N. Y.

P. B. PATTERSON, M.A.E. (Florida) Instructor, University of Florida, Gainesville, Fla.

G. W. PAYNE, Student, St. Mary's University of San Antonio, Texas.

W. L. PICKARD, Ph.D. (Peabody) Professor, State University of New York, State Teachers College, Paltz, N. Y.

F. R. POMILLA, M.S. (Fordham) Instructor, St. John's University, Brooklyn, N. Y.

H. L. POND, M.S. (George Washington) Physicist, David Taylor Model Basin, Washington, D. C.

H. L. RANDOLPH, B.A. (California) Grad. Asst., University of California, Los Angeles, Calif.

D. P. RICHARDSON, Ph.D. (Chicago) Professor, University of Arkansas, Fayetteville, Ark.

JACOB SAMOLOFF, B.A. (Toronto) Instructor, Lehigh University, Bethlehem, Pa.

W. C. SANGREN, Ph.D. (Michigan) Asst. Professor, Miami University, Oxford, Ohio.

HENRY SCHUTZBERGER, M.S. (Illinois) Research Mathematician, Sandia Laboratory, Albuquerque, N. M.

N. T. SEELY, JR., M.A. (Pennsylvania) Instructor, Arkansas A. M. & N. College, Pine Bluff, Ark.

W. D. SERBYN, Student, University of Chicago, Ill.

EDWARD SILVERMAN, Ph.D. (California) Asst. Professor, Kenyon College, Gambier, Ohio.

S. C. SIMONSON, M.A. (Minnesota) Instructor, University of North Dakota, Grand Forks, N. D.

JAMES C. SMITH, JR., M.S. (St. Louis) Instructor, University of Notre Dame, Ind.

J. K. STERRETT, M.A. (Kansas) Asso. Professor, Marshall College, Huntington, W. Va.

MARY R. STEUDLE, M.A. (Canisius) Teacher, Kensington High School, Buffalo, N. Y.

E. A. STURLEY, M.A. (Yale) Asst. Professor, Allegheny College, Meadville, Pa.

MATILDA B. THOMPSON, M.A. (Columbia) Asso. Professor, North Dakota Agricultural College, Fargo, N. D.

W. R. TRANSUE, Ph.D. (Lehigh) Professor, Kenyon College, Gambier, Ohio.

JANE A. UHRHAN, M.S. (Indiana) Instructor, Butler University, Indianapolis, Ind.

DUNSTAN VELESZ, M.S. (DePaul) Instructor, Quincy College, Ill

K. W. WESTON, Student, University of Wisconsin, Waukesha, Wis.

R. A. ZEGERS, M.S. (Fordham) Chairman, Department of Mathematics, St. Peter's College, Jersey City, N. J.

THE NINTH AND TENTH CARUS MONOGRAPHS

Two more Carus Mathematical Monographs will be available in the spring of 1950. They are Monograph 9: *The Theory of Algebraic Numbers* by Harry Pollard and Monograph 10: *The Arithmetic Theory of Quadratic Forms* by B. W. Jones.

Every member of the Association is entitled to one copy of each Carus Monograph at $1.75. Orders (preferably accompanied by remittance) for Monographs 9, 10 or others should be sent to: Mathematical Association of America, University of Buffalo, Buffalo 14, New York.

Additional copies of Monographs 9 and 10 for members and copies for nonmembers may be purchased at $3.00 from John Wiley and Sons, 440 Fourth Avenue, New York 16, N. Y. In the case of Monographs 1 to 8, orders should continue to be sent to the Open Court Publishing Company, LaSalle, Ill.

GEOGRAPHICAL BOUNDARIES OF THE SECTIONS OF THE ASSOCIATION

At the Columbus meeting, the Board of Governors voted that Sectional Governors shall hereafter be elected by a mail vote to be conducted by the Secretary-Treasurer of the Association. The Committee on Section Meetings was authorized to determine the geographical boundaries of each Section and to allocate members to Sections for voting purposes.

Accordingly the Committee on Section Meetings (H. M. Gehman, Fred Robertson and W. V. Parker, chairman) has assigned to each Section the areas listed below. The officers of each Section have been consulted and have agreed to their assignment. Individual members who wish exceptions made may appeal to the Committee through the Secretary-Treasurer of the Association.

It is understood that the membership of each Section consists of all members of the Association who express a desire to be enrolled as members of the Section. Hence, a member may regularly attend meetings of more than one Section. But for purposes of eligibility to hold office and to vote for officers of a Section, the membership of each Section of the Association shall be defined as consisting of all members of the Association living in the geographical areas indicated below.

Sections	*Territory Included in the Section*
ALLEGHENY MOUNTAIN	Pennsylvania (west of longitude 77°30′W), West Virginia
ILLINOIS	Illinois
INDIANA	Indiana
IOWA	Iowa
KANSAS	Kansas
KENTUCKY	Kentucky
LOUISIANA-MISSISSIPPI	Louisiana, Mississippi
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA	District of Columbia, Maryland, Virginia
METROPOLITAN NEW YORK	New Jersey (north of latitude 40°30′ N), New York (south of latitude 42° N)
MICHIGAN	Michigan
MINNESOTA	Minnesota, North Dakota; Manitoba, Saskatchewan
MISSOURI	Missouri
NEBRASKA	Nebraska, South Dakota
NORTHERN CALIFORNIA	California (except for the 14 counties included under Southern California), Hawaii, Nevada
OHIO	Ohio
OKLAHOMA	Arkansas, Oklahoma
PACIFIC NORTHWEST	Alaska, Idaho, Montana, Oregon, Washington; Alberta, British Columbia
PHILADELPHIA	Delaware, New Jersey (south of latitude 40°30′ N), Pennsylvania (east of longitude 77°30′ W)
ROCKY MOUNTAIN	Colorado, Utah, Wyoming
SOUTHEASTERN	Alabama, Canal Zone, Florida, Georgia, North Carolina, Puerto Rico, South Carolina, Tennessee
SOUTHERN CALIFORNIA	California (the counties of San Luis Obispo, Kings, Tulare, Inyo, and Mono and the counties to the south of these)
SOUTHWESTERN	Arizona, New Mexico, El Paso County in Texas
TEXAS	Texas (except El Paso County)
UPPER NEW YORK STATE	New York (north of latitude 42° N); Ontario, Quebec
WISCONSIN	Wisconsin
NEW ENGLAND REGION	Connecticut, Maine, Massachusetts, New Hampshire, Rhode Island, Vermont; New Brunswick, Newfoundland, Nova Scotia, Prince Edward Island

THE MAY MEETING OF THE WISCONSIN SECTION

The seventeenth annual spring meeting of the Wisconsin Section of the Mathematical Association of America was held at Lawrence College, Appleton, Wisconsin on Saturday, May 14, 1949. Professor A. C. Berry, Chairman of the Section presided at the morning and afternoon sessions.

About sixty-five persons were present, including the following thirty-three members of the Association: K. J. Arnold, R. H. Bardell, Leon Battig, L. J. Berner, A. C. Berry, W. W. Bigelow, R. H. Bing, H. J. Cohen, B. H. Colvin, Rev. L. A. V. DeCleene, H. P. Evans, Fannie Hopkins, C. C. Hsiung, R. C. Huffer, J. F. Kenney, R. E. Langer, C. C. MacDuffee, Morris Marden, Sister Mary Felice, J. R. Mayor, P. E. Meadows, Karl Menger, O. E. Overn, G. A. Parkinson, H. P. Pettit, Elizabeth Sokolnikoff, R. D. Specht, J. C. Stewart, J. V. Talacko, J. J. Van de Castle, J. I. Vass, R. D. Wagner, Louise A. Wolf.

At the business meeting held after the afternoon program, the following officers were elected for the coming year: Chairman, J. R. Mayor, University of Wisconsin; Secretary, Louise A. Wolf, University of Wisconsin in Milwaukee; Program Committee, E. G. Harrell, Platteville State Teachers College, J. C. Stewart, Lawrence College, R. D. Wagner, Marinette Extension Division of the University of Wisconsin. It was agreed to have the May 1950 meeting at Marquette University in Milwaukee.

The following papers were presented:

1. *The projective space*, by Professor Karl Menger, Illinois Institute of Technology.

2. *An equilateral distance*, by Professor R. H. Bing, University of Wisconsin.

This paper will be published in a later issue of the MONTHLY.

3. *Some applications of Newton's formula in finite differences*, by Professor H. P. Pettit, Marquette University.

It has frequently been observed that Newton's formula is valid for interpolation when all differences beyond the kth vanish. Here it was shown that where the differences do not vanish but follow certain definite laws, Newton's formula still provides valid and useful results. This was done by applying the forms of the binomial expansion, notably forms such as $\sum_n C_i a^i$, $\sum i_n C_i a^i$, etc., both in the case of the differences $\Delta^i u_0$, and further differences found by treating these as a sequence to be analyzed just as the set U_n. Applications were given to distribution of various types.

4. *Report on the investigation of the A.A.A.S. committee on the status of the sciences and mathematics*, by Professor R. C. Huffer, Beloit College.

5. *A freshman course for general students*, by Sister Mary Petronia, Mount Mary College, introduced by Sister Mary Felice, Mount Mary College.

LOUISE A. WOLF, *Secretary*

THE MAY MEETING OF THE ILLINOIS SECTION

The twenty-eighth annual meeting of the Illinois Section of the Mathematical Association of America was held at Bradley University, Peoria, Illinois, on Friday afternoon and Saturday forenoon, May 13–14, 1949. Professor S. F. Bibb, Chairman of the Section, presided at all meetings.

There were fifty-six in attendance including the following forty-four members of the Association: H. G. Ayre, R. A. Baumgartner, H. R. Beveridge, S. F. Bibb, A. H. Black, B. K. Brown, F. R. Brown, S. S. Cairns, Joseph Clare, E. G. H. Comfort, J. J. Corliss, Paul Cramer, Marjorie L. Croft, I. K. Feinstein, L. R. Ford, A. E. Gault, G. D. Gore, A. E. Hallerberg, E. W. Hellmich, Rose L. Hornacek, E. C. Kiefer, W. A. Lafferty, C. T. McCormick, W. C. McDaniel, A. W. McGaughey, E. B. Miller, C. N. Mills, G. E. Moore, M. G. Moore, Elsie C. Muller, Grace M. Nolan, F. S. Nowlan, Margaret Owchar, Roberta E. Presnell, Haim Reingold, T. E. Rine, E. W. Schreiber, H. W. Schwartz, M. Anice Seybold, H. A. Simmons, D. S. Snader, R. C. Stephens, A. T. Street, and Alice Wright.

At the business meeting Saturday forenoon the following officers were

elected for the coming year: M. G. Moore, Bradley University, Chairman; W. C. McDaniel, Southern Illinois University, Vice-Chairman; and E. C. Kiefer, Millikin University, Secretary. The 1950 meetings are to be held at Southern Illinois University, Carbondale, Illinois, on Friday afternoon and Saturday forenoon May 12–13.

Professor L. R. Ford made a progress report regarding encouragement of mathematics among high school students in the Chicago area. He reported several meetings of the committee, of which he is chairman. They plan a series of contests and finally a careful examination to determine the outstanding mathematical minds. Lack of adequate funds hinders the promotion of the plan, but sources of aid seem to be in sight. Several of the Chicago Universities and Colleges stand ready to offer scholarships to the winners. A resolution was passed, and a copy sent to Bradley University, thanking the university for the cooperation and hospitality extended to the Section.

A committee composed of L. R. Ford (Chairman), M. Anice Seybold, and C. T. McCormack was appointed to formulate plans to organize the Illinois Section under a constitution and by-laws. This committee is to report at next year's meeting in Carbondale.

The following program was given:

1. *So you are going to select a text-book in college algebra*, by Dr. J. J. Corliss, University of Illinois, Navy Pier Branch.
2. *The theorem of Coriolis*, by Dr. G. D. Gore, Roosevelt College of Chicago.
3. *The new interpretation of the "belonging to exponents,"* by Dr. H. A. Simmons, Northwestern University.
4. *The story of the Illinois Section*, by Dr. C. N. Mills, Illinois State Normal University.
5. *The Gibbs phenomenon in Fourier series*, by Dr. E. G. H. Comfort, Illinois Institute of Technology.
6. *The training of mathematical teachers*, by Dr. H. G. Ayre, Western Illinois State College.
7. *The teaching of the calculus from the teachers college point of view*, by Dr. E. W. Hellmich, Northern Illinois Teachers College.

E. C. KIEFER, *Secretary*

MAY MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The spring meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at the University of Virginia, Charlottesville, Virginia on Saturday, May 14, 1949. Mr. Michael Goldberg, Chairman of the Section, presided.

There were seventy-seven persons attending the meeting, including the following fifty-four members of the Association: M. W. Aylor, N. H. Ball, R. R. Bernard, Ruth E. Biggers, W. E. Bleick, J. W. Blincoe, T. A. Botts, R. S. Burington, H. H. Campaigne, Randolph Church, H. F. DeFrancesco, C. H.

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The following papers were presented:

1. *Application of matrix algebra to the solution of normal equations*, by Mr. J. L. Stearn, U. S. Coast and Geodetic Survey, introduced by the Secretary.

A non-Cayleyian matrix algebra was defined. A proof was then given of the Doolittle technique for solving normal equations when the coefficient matrix is positive definite. The usefulness of this matrix algebra for determining the inverse of a high order symmetrical matrix was indicated.

2. *Steady rotational plane flow of a gas*, by Professor M. H. Martin, University of Maryland.

By taking the stream lines and isobars for curvilinear coordinate lines, the problem of determining the steady, rotational plane flow of a gas is reduced to the integration of a quasi-linear partial differential equation in which the pressure p and the stream function ψ are the independent variables, and the unknown function is $\theta=\theta(p, \psi)$, the direction of flow. This partial differential equation involves the speed $q=q(p, \psi)$, and plays a dual role in the theory depending on whether q or θ is taken as a known function of p, ψ. In the latter case the partial differential equation becomes a linear partial differential equation of the third order for the unknown q. Flows for which the isobars are isoclines, $\theta=\theta(p)$, are excluded from the general theory and are treated separately. As an example of the inverse approach in which θ is preassigned, a detailed investigation is carried out for the special case $\theta=\theta(\psi)$ in which the stream lines are straight lines.

3. *Some remarks concerning the symbolics in vector analysis*, by Dr. M. M. Munk, Naval Ordnance Laboratory, introduced by Mr. M. A. Hyman.

The controversy about signs in vector analysis was discussed, and a few reasons for using Hamilton's sign rule were presented. Clarity and versatility may be improved by the use of Jaumann's symbol for dyadic multiplication, and by Jaumann's rule for the expansion of vector differentiation, and also by the use of a symbol for the space integral which is the inverse of the differentiation "nabla."

4. *Visual education—the slide rule*, by Professor W. F. Shenton, American University.

A lantern slide with various motions was demonstrated. By means of it settings of a slide rule were projected on a screen in size large enough to make explanation of the operation of a rule of the Mannheim or duplex type possible to a large number of students at one time.

5. *A theorem on local convexity*, by Mr. V. L. Klee, Jr., University of Virginia.

A subset X of a linear space is called "locally convex" if each of its points has a neighborhood whose intersection with X is convex. In 1928 it was proved independently by H. Tiete and S.

Matsumura that a closed, connected, locally convex subset of euclidean n space must actually be convex. In 1942 this result was extended to normed linear spaces by I. J. Schoenberg and M. Nakamura (independently). Mr. Klee presented a new proof of this theorem, simpler than its predecessors and valid in an arbitrary topological linear space (in the sense of Hille).

6. *The Russian peasants' method of multiplication*, by Mr. Sidney Kaplan, Naval Ordnance Laboratory.

The Russian peasants' method of multiplication was presented and compared with the Egyptian method.

7. *Non-digital computing devices in the teaching of mathematics*, by Professor Randolph Church of the U. S. Naval Postgraduate School.

A few devices for adding, multiplying, integrating, producing special functions and combinations of these for solving systems of linear equations and algebraic equations, for obtaining Fourier coefficients and for other special purposes, were very briefly described. These were selected to support the thesis that the mathematical explanation of such devices can provide illuminating and valuable illustrative material at various levels in the teaching of mathematics. Several of the instruments described were exhibited, including a Mader-Ott harmonic analyzer and an instrument for evaluating Stieltjes integrals designed by E. J. Nystrom.

FLORENCE M. MEARS, *Secretary*

CALENDAR OF FUTURE MEETINGS

International Congress of Mathematicians, Cambridge, Massachusetts, August 30–September 6, 1950.

Thirty-fourth Annual Meeting, University of Florida, Gainesville, December 30, 1950.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN

ILLINOIS, Southern Illinois University, Carbondale, May 12–13, 1950.

INDIANA, Wabash College, Crawfordsville, April 29, 1950.

IOWA, State University of Iowa, Iowa City, April 21–22, 1950.

KANSAS, State Teachers College, Pittsburg, April 22, 1950.

KENTUCKY, University of Kentucky, Lexington, April 29, 1950.

LOUISIANA-MISSISSIPPI, Centenary College, Shreveport, Louisiana, February 24–25, 1950.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, University of Richmond, Virginia, May, 1950.

METROPOLITAN NEW YORK, City College, April 1, 1950.

MICHIGAN, University of Michigan, March 25, 1950.

MINNESOTA, Macalester College, St. Paul, May 6, 1950.

MISSOURI, Washington University, St. Louis, March 24–25, 1950.

NEBRASKA, Nebraska Wesleyan University, Lincoln, May 6, 1950.

NORTHERN CALIFORNIA

OHIO, Denison University, Granville, April 22, 1950.

OKLAHOMA

PACIFIC NORTHWEST, University of Washington, Seattle, June, 1950.

PHILADELPHIA

ROCKY MOUNTAIN, University of Denver, April, 1950.

SOUTHEASTERN, University of Florida, Gainesville, April 7–8, 1950.

SOUTHERN CALIFORNIA, Immaculate Heart College, Hollywood, March 11, 1950.

SOUTHWESTERN, Spring, 1950.

TEXAS, Abilene, April 14–15, 1950.

UPPER NEW YORK STATE, Syracuse University, April 22, 1950.

WISCONSIN, Marquette University, Milwaukee, May 13, 1950.

ANALYTIC GEOMETRY

By *Lyman M. Kells* and *Herman C. Stotz, United States Naval Academy*

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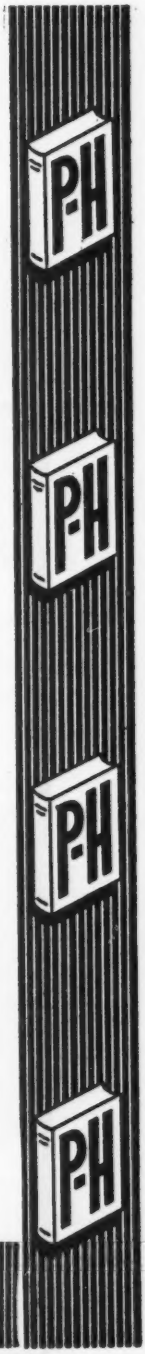

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